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FUNDAMENTALS OF BUSINESS MATHEMATICS

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SECOND EDITION

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PREFACE TO SECOND EDITION

In this second edition of *Fundamentals of Business Mathematics* several changes in content and order of presentation have been made, but the general plan of presentation has been retained. Data used in many examples and exercises have been brought up to date, and algebraic topics have been reduced to include only the essentials needed for mastery of the text material.

The success of the book during the past seven years has encouraged the authors to attempt to realize more fully the aims set forth in the preface to the first edition. Greater emphasis has been placed upon actual applications of mathematics in business, but ample opportunity is afforded for practice and the development of skills needed in making these applications.

In response to many requests by students, instructors, and persons engaged in business, the authors have placed increased emphasis upon the statistical aspects of business mathematics. The discussion of probability has been woven into the chapter on further statistical methods, and a section dealing with sampling and estimation has been added by way of introduction to these important applications of statistical methods in the business world.

The authors are grateful for the many helpful suggestions given by those who have used this text. They hope that the results of the efforts of all concerned will increase the usefulness of the text, and that this edition will, in part, express their thanks for the fine success that the first edition has enjoyed.

W. R. V. V.
C. W. T.

PREFACE TO FIRST EDITION

This book is designed to meet the needs of students of business. It aims to provide the kind and amount of mathematical background needed for success in the study of finance, commerce, accountancy, business statistics, and related topics. It is particularly adapted for use in schools and colleges of business administration, commerce, and finance, and in junior colleges.

The material in the book has been tested over a period of five years in classes of students majoring in business at Fenn College, where it was used in mimeographed form. As a result of the experience gained from the reactions of students and instructors, the authors have, after several revisions, included only those fundamentals upon which later study depends. The carefully selected topics are presented in a simple manner and lead the student gradually from one to another.

General principles are approached through simple illustrations. The student is then given opportunity to prove to himself that he understands developments by beginning with simple problems patterned after those illustrations, and continuing with more complicated problems which require intelligent use of material already covered. For example, the principles involved in the application of geometric progressions to problems of finance are first developed through carefully selected illustrations, and additional simple problems are suggested for practice. More than 3,000 drill and practical problems of graded difficulty are included in the text, and a self-test is given at the end of each chapter.

For students who need extra training or review in arithmetic and algebra, there is sufficient material in the first two chapters where the treatment is thorough from the most elementary beginnings. Those who are proficient in algebra may begin with the third chapter, Introduction to Statistics, and then continue with percentage, simple interest, and discount, and probability. The second half of the book begins with further algebraic operations as preliminary to the study of progressions, compound interest, annuities, and applications of equations in business problems, and ends with further statistical methods so important in the analysis of business data. The entire book can be covered in a one-year course of about 100 hours. For most groups of students the first semester should include the first six chapters and the second semester the remainder of the book.

The authors are grateful to their colleagues in the Department of Mathematics of Fenn College for their encouragement and helpful suggestions—especially to Professor M. B. Tolar, at whose suggestion the book has been written, to Miss Alberta Prasse, who was mainly responsible for the preparation of the mimeographed material after each revision, and to Professor K. D. Kelly, who read the proofsheets critically. They wish to thank their wives, June Van Voorhis and Jane Topp, for their assistance during the course of the book's writing, and are under special obligation to Dr. A. A. Bennett of Brown University, who made valuable suggestions for the final revision of the manuscript.

W. R. V. V.
C. W. T.

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Chapter I

SIMPLE ARITHMETIC OPERATIONS

Introduction. In every field of modern business there are many problems that must be solved by the use of arithmetic or elementary algebra. In fact, the operation of certain phases of many businesses depends primarily upon the application of some phase of mathematics. The computation of interest on loans, the determination of profit on investment, the measurement of the effectiveness of different advertising media, the calculation of monthly pay rolls, and the keeping of records of student grades in the office of the college registrar are but a few examples of the multitude of situations where the fundamentals of business mathematics are applied.

The skills required for the successful performance of arithmetic and algebraic operations can be developed through review and practice. For this reason, the first two chapters are devoted to some of the fundamental elements of arithmetic and algebra needed in business mathematics. Other phases of these topics are taken up later in the book as they are needed.

From a logical point of view, one might wish to begin the discussion of arithmetic operations with the real number system and examine some of the properties of real numbers. However, since most persons are probably familiar with the simpler aspects of our number system, we shall begin with a review of the fundamental operations with positive numbers.

1. Addition. In addition, numbers may be arranged vertically in columns with respect to the decimal point. We may add one column at a time, taking each digit as we come to it, or we may group the figures as we proceed through the column. In some instances we may find it convenient to add more than one column at a time. In short, the method we use depends largely upon the nature of the column of numbers to be added. Consider, for example, the following illustrations:

(*a*)

$$\begin{array}{r} 321.63 \\ 32.97 \\ 105.12 \\ 7.84 \\ \underline{0.50} \\ 468.06 \end{array}$$

Starting at the bottom of the first column of (*a*), we may say mentally, "0, 4, 6, 13, 16." Write 6, and carry 1. Continuing downward through the second column, "7, 16, 17, 25, 30." Put down 0 and carry 3, and so forth. We could have begun, of course, at the top of the first column and then continued upward through the second column, or we could have treated each column in the upward or downward direction, as we preferred.

If we prefer to group numbers as we add, we might begin at the top of the first column, taking the 3 and 7 together, saying "10," and then, upon grouping the 2 and 4, saying "16." Many persons, however, prefer to add each digit as they come to it.

(*b*)

$$\begin{array}{r} 6 \\ 15 \\ 10 \\ \underline{25} \\ 56 \end{array}$$

In adding (*b*), it might be desirable to add the two columns at one time. Starting at the bottom, we could say, "25, 35, 50, 56." On the other hand, we might prefer to accumulate the tens column first and then include the units column, saying, "20, 30, 40, 45, 50, 56," again reading from the bottom.

In horizontal addition the numbers are connected by *plus* signs and are added without arranging them in a column. It is well to become adept at this sort of adding because it quite frequently eliminates the necessity of recopying numbers into columns. Care must be taken with the decimal point so that no error is made in adding the corresponding digits of the numbers.

(*c*) $23.27 + 13.54 + 22.76 + 106.89 + 5.29 = 171.75.$

Starting at the right-hand digit of the last number to be added in (*c*), we add the corresponding digits, saying mentally, "9, 18, 24, 28, 35." Put down 5 and carry 3. Then for the next corresponding digits, say "5, 13, 20, 25, 27"; put down 7 and carry 2. This process is continued for all sets of corresponding digits.

PROBLEMS

1. Find the sum by adding one column at a time, taking each digit as you come to it.

(a)	(b)	(c)	(d)	(e)
20.63	112.26	129	32,945.24	23,125,218.37
12.17	15.37	348	95,472.53	63,294,146.07
92.39	86.17	265	6,195.29	80,217,729.23
14.65	137.59	521	2,948.15	62,102,275.15
27.84	209.24	659	7,800.62	86,837,325.76
56.19	8.73	196	10,763.97	24,453,479.98
63.76	126.87	843	23,801.03	22,092,468.08
29.35	13.72	381	90,765.25	98,723,554.55

2. Work the above problems by grouping figures as you proceed through the columns.

3. Try the method of picking out groups of ten as you proceed through the columns. [Starting at the bottom of the first column in (*a*), think 10 as a part of the sum of 5 and 6, and visualize the excess 1 as combining with the 9 to make another 10, and say "20," and so forth.]

4. Find each of the following sums without arranging in vertical columns:

(*a*) 27 + 314 + 129 + 684 + 27 + 960 + 395 + 724,
(*b*) 327.45 + 850.45 + 963.74 + 63.86 + 300.27 + 362.91,
(*c*) 64.257 + 961.360 + 286.317 + 10.587 + 619.035 + 900.572,
(*d*) 457.36 + 589.84 + 6,904.75 + 313,829.10 + 16,946.87.

5. Complete the following sales record. Check by adding both vertical and horizontal totals.

Salesman	*Monday*	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*	*Total*
A	$165.35	$193.27	$112.59	$ 98.55	$107.94	$208.59	$______
B	87.28	123.93	111.83	84.80	97.63	119.98	______
C	97.37	153.95	88.87	103.35	76.67	99.10	______
D	126.49	95.50	125.25	115.15	85.45	76.85	______
E	77.56	63.82	79.56	91.10	77.99	63.47	______
F	89.63	97.95	96.29	126.04	75.50	129.58	______
Total	$	$	$	$	$	$	$

6. The following list of exercises should require no more than five minutes. Work the entire list while timing yourself. If it takes you longer than five minutes, practice the methods suggested in the preceding exercises, and repeat, working the exercises below until you have attained the desired degree of proficiency:

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
332	375	280	251	660
107	277	76	6,235	699
209	648	757	687	7,196
888	349	606	2,915	5,673

(*f*)	(*g*)	(*h*)	(*i*)	(*j*)
3,046	2,632	1,076	2,073	207
5,839	566	6,074	214	86
7,215	492	2,625	8,731	509
7,643	3,007	3,538	640	8,176
2,073	445	5,368	1,257	2,064

(*k*)	(*l*)	(*m*)	(*n*)	(*o*)
237.41	10,076.8	226.35	7.63	2,503.61
652.28	3,096.3	2,567.55	8.01	221.90
392.27	654.7	505.95	.74	63.12
6,182.96	5,079.8	107.58	.99	9.00

2. Subtraction. Probably most persons find the difference between two numbers by placing the subtrahend underneath the minuend and obtaining the difference column by column, "borrowing" from the column of next higher order whenever necessary. Thus in example (*a*) one could say, "6 from 8 leaves 2, 9 from 16 leaves 7, 1 from 1 leaves 0, and 3 from 7 leaves 4."

(*a*)
7,268
3,196
4,072

This method of borrowing involves a mastery of all the subtraction combinations, just as addition involves the addition combinations. Thus, corresponding to 7 *plus* 9 = 16 we must recognize immediately 16 *minus* 9 = 7 if we wish to perform the subtraction with speed and accuracy.

Many errors in subtraction can be traced to the failure on the part of the student to remember that a digit has been reduced by the borrowing process. Thus he might forget that after saying, "9 from 16 leaves 7," the next subtraction must be "1 from 1," and not "1 from 2." These errors can be eliminated by making subtraction a process of addition. Consider example (*b*). Instead of "borrowing," as above, say "9 and 7 is 16." Write the 7, then add the 1 to the 5 of the subtrahend, and say "6 and 6 is 12." Write the 6, then add the 1 to the 8 of the subtrahend, and say, "9 and 4 is 13." Write the 4, then add the 1 to the 4 of the subtrahend, and say, "5 and 3 is 8." Write the 3. The subtraction may, of course, be indicated by the *minus* sign: 8,326 − 4,859 = 3,467.

$$\begin{array}{r} (b) \\ 8{,}326 \\ \underline{4{,}859} \\ 3{,}467 \end{array}$$

PROBLEMS

1. Practice the "addition" method in performing the following subtractions:

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
3,214	56,934	6,209.56	6,283,986	79.39
2,947	24,834	1,693.75	5,103,059	37.48

2. Find the following differences without recopying the numbers:

(*a*) 67 − 32, (*e*) 473 − 238, (*i*) 78.62 − 19.88,
(*b*) 39 − 27, (*f*) 794 − 596, (*j*) 10.34 − 8.61,
(*c*) 86 − 48, (*g*) 6,643 − 3,957, (*k*) 799.84 − 555.55,
(*d*) 71 − 27, (*h*) 395,593 − 139,685, (*l*) 2,085.38 − 1,997.69.

3. What amount of change is to be given from $5 for each of the following purchases? (*a*) $3.75, (*b*) $2.30, (*c*) $1.65, (*d*) $1.45, (*e*) $4.87, (*f*) $3.16, (*g*) $4.72, (*h*) $2.28, (*i*) $2.32, (*j*) $0.13, (*k*) $3.91, (*l*) $1.58.

4. What amount of change is to be given from $10 for each of the following purchases? (*a*) $2.59, (*b*) $4.17, (*c*) $6.66, (*d*) $8.23, (*e*) $3.78, (*f*) $5.82, (*g*) $7.35, (*h*) $9.44, (*i*) $1.27, (*j*) $2.33, (*k*) $3.48, (*l*) $4.56.

5. Complete the following sales summary. (Sales less cost equals gross profit; gross profit less expenses equals net profit.)

Item	*Sales*	*Cost*	*Gross Profit*	*Expenses*	*Net Profit*
A	$397.27	$376.79	$	$ 4.16	$
B	937.46	711.38		96.43	
C	892.16	529.85		78.96	
D	663.48	552.05		93.37	
E	734.23	611.61		82.62	
Totals	$	$	$	$	$

6. The following table shows the per-capita receipts and expenditures of the Federal government during World War II. Complete the deficit column, and find the totals. Explain the smaller deficits for 1942 and 1946.

Year	*Receipts*	*Expenditures*	*Deficit*
1942	$ 95.25	$241.80	
1943	163.25	572.83	
1944	319.92	679.29	
1945	332.78	719.25	
1946	305.25	461.17	
Totals	$	$	$

7. The following list of exercises should require no more than three minutes. Work the entire list while timing yourself. If it takes you longer than three minutes, practice the "addition" method, and repeat the exercises until you have attained the desired degree of proficiency. Subtract:

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
367,938	845,295	376,468	657.84	653,896,396
153,694	537,692	226,999	339.77	497,396,290

(*f*)	(*g*)	(*h*)	(*i*)	(*j*)
4,985,908	985,043	935,739	489,756	8,945,678
3,785,444	668,833	777,999	233,355	5,666,888

(*k*)	(*l*)	(*m*)	(*n*)	(*o*)
555,323.71	65,444.35	83,684	2,568.856	684.9567
263,578.68	27,864.56	38,957	1,005.357	264.4738

3. Multiplication. Multiplication of positive whole numbers is simply a shortened process of addition. To multiply 4,328 by 562, for example, we could write 4,328 down 562 times in a column and then add the column. This would be a waste of time. The shortened process makes use of the fact that multiplying by 562 is equivalent to multiplying by 2, then by 60, then by 500, and then adding together the partial products. Thus we write

4328	(Multiplicand)	4328	
562	(Multiplier)	562	
8656	Multiplying by 2	8656	(Omitting unnecessary zeros)
259680	Multiplying by 60	25968	
2164000	Multiplying by 500	21640	
2432336	(Product)	2432336	

Using the multiplication sign, ×, the product may be written $562 \times 4{,}328 = 2{,}432{,}336$.

A number of short cuts in multiplication are of practical value if they are mastered. Most of these are self-evident, and many will suggest themselves if one will take the time to familiarize oneself with multiplication number combinations. A few of the most useful short cuts are:

To multiply by 10, move the decimal point one place to the right; by 100, two places to the right; by 1,000, three places, and so forth.
To multiply by 25, multiply by 100 and divide by 4.
To multiply by 50, multiply by 100 and divide by 2.
To multiply by 99, multiply by 100 and subtract the number.
To multiply by 101, multiply by 100 and add the number.
To multiply by 75, multiply by 300 and divide by 4.

A useful short cut in multiplying a number by 11 is evident from an illustration. To multiply 693 by 11, proceed as follows:

3 (Right-hand digit of 693)

$9 + 3 = 12$ (Sum of two right-hand digits of 693)

$6 + 9 + 1 = 16$ (Sum of digits 6 and 9, plus 1 carried over from the 12)

$6 + 1 = 7$ (Sum of first digit, plus 1 carried over from 16)

The answer is 7,623. This method requires no written computation.

PROBLEMS

1. Multiply mentally:

(*a*) 240 × 25,
(*b*) 356 × 99,
(*c*) $23.84 × 50,
(*d*) 15 × 101,
(*e*) 56 × 21,
(*f*) 75 × 8,
(*g*) 326.49 × 10,
(*h*) 752.35 × 100,
(*i*) $0.85 × 1,000,
(*j*) 32 × 40,
(*k*) 7.24 × 500,
(*l*) 250 × 98,
(*m*) 23.16 × 250,
(*n*) 75 × 9,
(*o*) 250 × 102,
(*p*) 124 × 75,
(*q*) 420 × 51,
(*r*) 3,273 × 50,
(*s*) 972 × 11,
(*t*) 6,354 × 11,
(*u*) 87,314 × 11,
(*v*) 90,721 × 11,
(*w*) 1,613,962 × 11.

2. Visualize the partial products, and write the answers only:

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)	(*f*)
32	53	76	65	66	49
24	25	89	27	88	57

3. Find the total pay roll:

Number of Employees	*Hours Worked*	*Rate per Hour*	*Amount*
57	43	$1.85	$_____
98	45	1.80	_____
102	44	1.87	_____
49	40	1.95	_____
125	44	1.82	_____
74	45	1.85	_____

4. The following list of multiplication exercises should require no more than four minutes. Work the entire list while timing yourself. (The student should practice the exercises until he is able to complete them in the given time.)

(*a*)	(*b*)	(*c*)	(*d*)	(*e*)
61,429	5,973	42,854	8,943	32,741
184	69	256	68	320

(*f*)	(*g*)	(*h*)	(*i*)	(*j*)
7,368	1,604	94,085	72,396	749
859	2,068	327	58	84

5. Multiply mentally the following numbers by 11: 56, 217, 489, 3.46, 1.06, 2,673.

6. Multiply mentally the following numbers by 25, 50, 99, and 101: 44, $450, $4.60, 1.035, $150.20, 0.007.

4. Division. Division may be thought of as a shortened process of subtraction. When we write $39 \div 13 = 3$, or $13\overline{)39} = 3$, we imply that 39 contains 13 three times. The quotient 3 represents the number of times 13 can be taken out of 39 by continued subtraction:

$$\begin{aligned} 39 - 13 &= 26, \\ 26 - 13 &= 13, \\ 13 - 13 &= 0. \end{aligned}$$

It is evident that this process would ordinarily be too lengthy to be practical. The method in common use is essentially a rapid way of performing the continued subtractions and is obviously the inverse of multiplication. The following example will illustrate the familiar process:

$$\begin{array}{rrl} & 14 & \text{(Quotient)} \\ \text{(Divisor)} & 375\overline{)5287} & \text{(Dividend)} \\ & \underline{375} & \\ & 1537 & \\ & \underline{1500} & \\ & 37 & \text{(Remainder)} \end{array}$$

Check:

$375 \times 14 = 5{,}250.$
$5{,}250 + 37 = 5{,}287.$

PROBLEMS

1. Perform the following divisions, and check your results: (*a*) $364\overline{)94{,}572}$, (*b*) $5{,}275\overline{)328{,}408}$, (*c*) $352\overline{)69{,}302}$, (*d*) $627\overline{)10{,}010}$, (*e*) $903\overline{)68{,}320}$.

2. The following device is very useful in practicing simple short division. Divide the first dividend by the first divisor, and then use the quotient thus obtained as the dividend for the next division. The final quotient should be the same for each column.

(*a*)	(*b*)	(*c*)	(*d*)
$2\overline{)4,354,560}$	$3\overline{)4,354,560}$	$9\overline{)4,354,560}$	$9\overline{)4,354,560}$
$3\overline{)\qquad}$	$5\overline{)\qquad}$	$7\overline{)\qquad}$	$8\overline{)\qquad}$
$4\overline{)\qquad}$	$7\overline{)\qquad}$	$5\overline{)\qquad}$	$7\overline{)\qquad}$
$5\overline{)\qquad}$	$9\overline{)\qquad}$	$3\overline{)\qquad}$	$6\overline{)\qquad}$
$6\overline{)\qquad}$	$2\overline{)\qquad}$	$8\overline{)\qquad}$	$5\overline{)\qquad}$
$7\overline{)\qquad}$	$4\overline{)\qquad}$	$6\overline{)\qquad}$	$4\overline{)\qquad}$
$8\overline{)\qquad}$	$6\overline{)\qquad}$	$4\overline{)\qquad}$	$3\overline{)\qquad}$
$9\overline{)\qquad}$	$8\overline{)\qquad}$	$2\overline{)\qquad}$	$2\overline{)\qquad}$

3. The student should make certain that he can complete the following list of division exercises in five minutes or less. (*a*) $23\overline{)6,430}$, (*b*) $206\overline{)1,372,074}$, (*c*) $37\overline{)40,327}$, (*d*) $65\overline{)67,839}$, (*e*) $39\overline{)37,497}$, (*f*) $60\overline{)47,892}$, (*g*) $678\overline{)678,356}$, (*h*) $501\overline{)601,368}$, (*i*) $93\overline{)39,750}$, (*j*) $43\overline{)100,574}$.

5. Factoring a Positive Integer.

Quite frequently in carrying out arithmetic computations it is desirable to express a number as the product of two or more numbers. We may, for example, wish to write the number 30 as 3×10 or as 2×15 or as 6×5 or as $2 \times 3 \times 5$. Of course, we could include the number 1 with each of these groupings and write $3 \times 10 \times 1$, $2 \times 15 \times 1$, and so forth, since it is obvious that every number can be written as the product of itself and 1. Each of the numbers 2, 3, 5, and 10 (as well as 30 and 1) is called a *factor* of the number 30. It is seen that each factor is an *exact* divisor of 30. In general, we say that a factor of a number is any exact divisor of a number.

A number that is divisible only by itself and 1 is called a prime number. Thus, 2, 3, 5, 7, 11, 13, 17, and 19 are the prime numbers between 1 and 20. The factors of a number which are prime are called the *prime factors* of that number. Thus, the prime factors of 30 are 2, 3, and 5, the number 1 being excluded.

The process of expressing a number as the product of its factors is called *factoring*. There is little need to memorize a set of rules to be followed in performing this process, if one will remember such simple truths as the following:

(*a*) Every number ending in 0, 2, 4, 6, 8 is divisible by 2.

(*b*) A number is divisible by 3 if the sum of the digits is a multiple of 3.

(*c*) A number is divisible by 4 if the number represented by the last two digits is divisible by 4.

(*d*) Numbers ending in 5 or 0 are divisible by 5.

(*e*) A number is divisible by 9 if the sum of the digits is a multiple of 9.

Other number properties will suggest themselves as the student practices factoring numbers.

Illustrations:

1. Find the prime factors of 420.

We observe that 420 is divisible by 4. Write 4 as 2 × 2. Then divide 105 by 5. Continue the process until all prime factors are found.

$$\begin{array}{rr} 2 \times 2 & 4\underline{)420} \\ 5 & 5\underline{)105} \\ 7 & 7\underline{)\ 21} \\ 3 & 3 \end{array}$$

The prime factors of 420 are, therefore, 2, 2, 5, 7, 3—that is, $2 \times 2 \times 5 \times 7 \times 3 = 420$.

2. Find the prime factors of 3,528.

$$\begin{array}{rr} 2 \times 2 \times 2 & 8\underline{)3528} \\ 3 \times 3 & 9\underline{)441} \\ 7 & 7\underline{)49} \\ 7 & 7 \end{array}$$

The prime factors of 3,528 are, therefore, 2, 2, 2, 3, 3, 7, 7—that is, $2 \times 2 \times 2 \times 3 \times 3 \times 7 \times 7 = 3{,}528$.

PROBLEMS

1. Find the prime factors of (*a*) 168, (*b*) 300, (*c*) 363, (*d*) 480, (*e*) 784, (*f*) 1,512, (*g*) 1,125, (*h*) 884, (*i*) 722.

2. Write the prime numbers between 1 and 100.

3. State the prime factors of 35, 21, 77, 121, 49.

4. The numbers 3, 5, 5, 7, and 11 are the prime factors of what number?

5. What prime factor is common to the two numbers 360 and 2,695?

6. Name two prime factors common to 30 and 48.

7. Which of the following numbers have no factors except 1 and themselves: 4, 5, 6, 7, 10, 11, 12, 13?

8. From this list select the numbers divisible by 2: 4,907, 660, 6,235, 4,027, 2,356, 37,264.

9. From this list select the numbers divisible by 3: 2,467, 6,075, 9,636, 8,124, 1,872, 12,475.

10. From this list select the numbers divisible by 5: 1,835, 6,778, 3,100, 2,348, 4,345, 21,035.

11. From this list select the numbers divisible by 9: 1,001, 5,004, 9,009, 6,264, 2,244, 12,342.

6. Greatest Common Divisor. Two rolls of curtain material contain 360 yd and 300 yd, respectively. The sales manager desires to cut the material of both rolls into pieces of equal length. It is desired, further, that these equal pieces be as great as possible. What should be the length of each piece?

It is seen that the length of the piece decided upon must, if possible, be such that it will divide an exact number of times into both 360 and 300. It could be 1 yd long or 2 yd or 3 yd long or any of a number of lengths; but we must find the *greatest length* into which the material could be cut and still obtain an exact number of pieces from each roll. That is to say, we desire the *greatest common divisor* of the numbers 360 and 300. How shall we find this number?

The greatest common divisor (written GCD) of the numbers 360 and 300 must contain as many prime factors of each magnitude as are common to both of these numbers. If, therefore, the prime factors of the two numbers are written, we should be able to obtain by inspection the GCD by making up a number containing every prime factor the least number of times it appears in each number. Thus,

$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5,$$

$$300 = 2 \times 2 \times 3 \times 5 \times 5.$$

The prime factor 2 appears *twice* in 300 and three times in 360; the prime factor 3 appears *once* in 300 and twice in 360; finally, the prime factor 5 appears *once* in 360 and twice in 300. The number we seek, therefore, will be the product of 2 taken twice, 3 taken once, and 5 taken once. Thus, the greatest common divisor of 360 and 300 equals $2 \times 2 \times 3 \times 5 = 60$. The two rolls of curtain material can be cut as desired into pieces 60 yd long.

The greatest common divisor of more than two numbers may be found in a similar way. It is simply the largest number that "goes into" each of the numbers exactly.

To find the GCD of two or more numbers, express each number as the product of its prime factors, and then take all the prime factors common to all the numbers the least number of times they appear as factors of any one number. The product of the prime factors selected is the GCD.

Illustration: Find the GCD of the numbers 60, 96, and 180.

$$\begin{aligned} 60 &= 2 \times 2 \times 3 \times 5, \\ 96 &= 2 \times 2 \times 2 \times 2 \times 2 \times 3, \\ 180 &= 2 \times 2 \times 3 \times 3 \times 5. \end{aligned}$$

The prime factors *common* to all three numbers are 2 and 3. Since 2 appears twice the least number of times and 3 appears once the least number of times, the GCD is $2 \times 2 \times 3 = 12$. Notice that the 5 does not enter into the GCD since it is not common to all three numbers.

PROBLEMS

1. Find the GCD of the following sets of numbers: (*a*) 72, 90, and 54; (*b*) 96, 120, and 360; (*c*) 152 and 368; (*d*) 245 and 875; (*e*) 12, 400, and 9,800; (*f*) 726, 1,210, and 440.

2. A plot of ground measures 35 by 42 by 77 by 140 ft. What is the greatest length of board that can be used in fencing the plot without cutting any of the boards?

3. Find the greatest common divisor of 75 and 100.

4. In reducing $\frac{24}{32}$ to lowest terms, is it sufficient to cancel the common divisor 2? What kind of divisor must be canceled in reducing a fraction to lowest terms?

5. Find the greatest common divisor of 9, 15, 21, and 60.

6. An agent having on hand three balls of rope containing 1,680, 1,584, and 3,120 ft, respectively, decides to sell the rope as clotheslines. What length must he decide upon to obtain the longest clotheslines without any waste?

7. Least Common Multiple. Units of tile blocks are formed by placing together 6-in. tile blocks; similarly with 8-in. and 10-in. tile blocks. What is the length of the shortest common unit that can be formed by using each of the three sizes of blocks?

The length of the unit of blocks must be such that it is exactly divisible by 6, 8, and 10. It must be, therefore, a *common multiple* of all three of these numbers. But since there are many numbers that are common multiples of 6, 8, and 10, our task will be to find the smallest of these common multiples, or the *least common multiple.*

The least common multiple (written LCM) of the numbers 6, 8, and 10 must contain as many prime factors of each magnitude as are found in any one of the numbers. The smallest number that will be exactly divisible by each of the three numbers will be found by multiplying together each different prime factor the greatest number of times it appears in any one of the numbers. Thus,

$$6 = 2 \times 3$$
$$8 = 2 \times 2 \times 2$$
$$10 = 2 \times 5$$

We must use the prime factor 2 three times in order to be assured that 8 will be a divisor of the LCM. The factor 3 must appear once, and the factor 5 must appear once, in order to have 6 and 10 as exact divisors. Thus, the LCM of 6, 8, and 10 equals $2 \times 2 \times 2 \times 3 \times 5 = 120$. The shortest common unit that can be formed by using each of the three sizes of tile blocks is 120 in. long.

The least common multiple of two or more numbers is simply the smallest number that will be exactly divisible by each of the several given numbers.

To find the LCM of two or more numbers, express each number as the product of its prime factors, and then take all the prime factors the greatest number of times they appear as factors of any one number. The product of the prime factors selected is the LCM.

Illustration: Find the LCM of the numbers 14, 18, and 300.

$$14 = 2 \times 7,$$
$$18 = 2 \times 3 \times 3,$$
$$300 = 2 \times 2 \times 3 \times 5 \times 5.$$

The greatest number of times that the prime factor 2 appears is twice (in the factors of 300). Likewise, the factor 3 appears twice (in the factors of 18), 5 twice (in the factors of 300), and 7 once (in the factors of 14). The LCM is $2 \times 2 \times 3 \times 3 \times 5 \times 5 \times 7 = 6{,}300$.

PROBLEMS

1. Find the LCM of the following sets of numbers: (*a*) 24 and 54; (*b*) 72, 48, and 90; (*c*) 121, 55, and 132; (*d*) 15, 25, 30, and 35; (*e*) 34, 20, and 66; (*f*) 48, 120, 36, and 60.

2. Find the LCM of (*a*) 6, 10, 15, 18; (*b*) 63, 12, 84, 7; (*c*) 54, 81, 63; (*d*) 8, 12, 20, 24, 25; (*e*) 2, 3, 4, 5, 6, 7, 8, 9; (*f*) 21, 64, 72, 18, 98.

3. Why is it that $0.50 is the smallest sum that can be paid by using either all nickels, all dimes, or all quarters?

4. What is the smallest container that may be filled by emptying full 2-qt, 2-gal, or 5-gal measures?

8. Common Fractions. A fraction is one or more of the equal parts into which a unit has been divided. For example, in expressing 2 ft as a part of a yard, we might think of the yard as being made up of three equal parts (3 ft) and write the fraction $\frac{2}{3}$. The number below the line (3), the *denominator*, indicates the number of equal parts into which the yard is divided. The number above the line (2), called the *numerator*, indicates the number of equal parts expressed. If we thought of the yard in terms of inches, there would be 36 equal parts, and 2 ft would represent 24 of these equal parts. The resulting *common fraction* would be $\frac{24}{36}$.

For convenience, common fractions, other than integers, may be classified as *proper* or *improper*. A proper fraction is one whose numerator is less than its denominator and represents, therefore, only a portion of a unit. Examples of proper fractions are $\frac{3}{4}$, $\frac{2}{3}$, $\frac{5}{8}$, $\frac{12}{18}$, and so forth. An improper fraction represents more than enough equal parts to make up a whole unit. Examples of improper fractions are $\frac{5}{4}$, $\frac{8}{3}$, $\frac{23}{6}$, $\frac{125}{50}$, and so forth.

When an improper fraction is expressed as a whole number and a fraction, it is called a *mixed number*. Thus, $1\frac{1}{4}$, $2\frac{2}{3}$, $4\frac{3}{5}$, $2\frac{25}{30}$, and so forth are examples of mixed numbers. It may be observed that improper fractions may be converted into mixed numbers by dividing the numerator by the denominator. Conversely, mixed numbers may be expressed as improper fractions by multiplying the whole number by the denominator of the fraction and then adding the numerator to the product for a sum which is the numerator of the improper fraction. Thus, to express $5\frac{3}{4}$ as an improper fraction, we calculate "4 times 5, plus 3," or 23, as the numerator and write $\frac{23}{4}$.

A fraction whose numerator and denominator contain one or more common factors may be reduced to a fraction of equivalent value expressed in "lower terms" by dividing both numerator and denominator by the common factor or factors. For example, $\frac{48}{60}$ can be written $\frac{4 \times 12}{5 \times 12}$, which, upon dividing both numerator and denominator by 12, we may reduce to $\frac{4}{5}$.

The process of dividing both terms of a fraction by the same number is called *cancellation*. It may be used to save much of the labor of long multiplication in problems involving a series of multiplications and divisions. For example,

$$\frac{\overset{2}{\cancel{14}} \times \overset{4}{\cancel{16}} \times \overset{\cancel{5}}{\cancel{125}} \times 11}{\cancel{4} \times \cancel{7} \times \cancel{25} \times \underset{3}{\cancel{15}}}, = \frac{88}{3} = 29\frac{1}{3}$$

9. Addition and Subtraction of Fractions. Fractions having the same denominator may be added or subtracted by adding or subtracting the numerators and placing the result over the common denominator. For example,

$$(a)\ \frac{3}{8} + \frac{2}{8} + \frac{5}{8} = \frac{10}{8},$$

$$(b)\ \frac{23}{12} - \frac{9}{12} = \frac{14}{12}.$$

If the denominators of the fractions to be added or subtracted are different, the fractions must first be changed so as to have the same denominator.

Thus,

$$(c)\ \frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6} = \frac{4+5}{6} = \frac{9}{6} = \frac{3}{2} = 1\frac{1}{2},$$

$$(d)\ \frac{7}{8} - \frac{1}{6} + \frac{2}{3} = \frac{21 - 4 + 16}{24} = \frac{33}{24} = 1\frac{9}{24} = 1\frac{3}{8},$$

$$(e)\ 5 + \frac{3}{7} = \frac{35}{7} + \frac{3}{7} = 5\frac{3}{7}.$$

If mixed numbers are to be added or subtracted, there are two methods that may be used. By the one method, all mixed numbers are changed to improper fractions and added or subtracted as above. Thus,

$$\begin{aligned}(f)\ 9\frac{2}{13} + 16\frac{2}{5} - 12 &= \frac{119}{13} + \frac{82}{5} - \frac{12}{1} \\ &= \frac{(119 \times 5) + (82 \times 13) - (12 \times 65)}{65} \\ &= \frac{595 + 1{,}066 - 780}{65} = \frac{881}{65} = 13\frac{36}{65}.\end{aligned}$$

By the other method, the whole-number parts of the mixed numbers and the fractional parts may be added or subtracted separately and then the two sums added or subtracted for the final answer. Thus, using example (*f*) above, we have

$$\begin{aligned}(g)\ 9 + 16 - 12 &= 13, \\ \frac{2}{13} + \frac{2}{5} &= \frac{10 + 26}{65} = \frac{36}{65}, \\ 13 + \frac{36}{65} &= 13\frac{36}{65}.\end{aligned}$$

PROBLEMS

1. Find the value of:

(*a*) $\frac{2}{5} + \frac{7}{10} + \frac{3}{20}$,
(*b*) $\frac{5}{8} + \frac{7}{12} + \frac{15}{24}$,
(*c*) $3\frac{1}{7} + 9\frac{5}{14} + \frac{8}{21}$,
(*d*) $16 - \frac{3}{8} + \frac{2}{5}$.

2. Reduce to their lowest terms:

(*a*) $\frac{12}{18}$,
(*b*) $\frac{30}{45}$,
(*c*) $\frac{60}{150}$,
(*d*) $\frac{126}{198}$,
(*e*) $\frac{615}{915}$,
(*f*) $\frac{120}{125}$,
(*g*) $\frac{777}{1998}$,
(*h*) $\frac{585}{1287}$,

3. Reduce to lowest terms: (*a*) $\frac{125}{625}$; (*b*) $\frac{128}{256}$; (*c*) $\frac{198}{165}$; (*d*) $\frac{1428}{2040}$; (*e*) $\frac{210}{315}$; (*f*) $\frac{450}{27}$; (*g*) $\frac{325}{10}$; (*h*) $\frac{1058}{2116}$.

4. Add:

(*a*) $\frac{4}{87}$, $\frac{17}{87}$, $\frac{48}{87}$, $\frac{76}{87}$, and $\frac{29}{87}$,
(*b*) $\frac{19}{144}$, $\frac{29}{144}$, $\frac{101}{144}$, and $\frac{11}{144}$,
(*c*) $\frac{6}{32}$, $\frac{15}{32}$, $\frac{18}{32}$, $\frac{12}{32}$, and $\frac{19}{32}$,
(*d*) $\frac{3}{4}$ and $\frac{5}{6}$,
(*e*) $\frac{5}{9}$ and $\frac{1}{9}$,
(*f*) $\frac{1}{4}$, $\frac{7}{8}$, and $\frac{11}{12}$,
(*g*) $2\frac{1}{2}$ and $3\frac{1}{6}$,
(*h*) $\frac{4}{5}$, $8\frac{3}{4}$, and $7\frac{1}{2}$,
(*i*) $16\frac{2}{3}$, $12\frac{3}{4}$, $2\frac{1}{4}$, and $8\frac{3}{5}$.

5. Subtract: (*a*) $\frac{5}{9}$ from $\frac{8}{9}$; (*b*) $\frac{7}{16}$ from $\frac{15}{16}$; (*c*) $\frac{1}{3}$ from $\frac{3}{4}$; (*d*) $\frac{1}{4}$ from $\frac{3}{5}$; (*e*) $\frac{2}{9}$ from $\frac{1}{4}$; (*f*) $4\frac{1}{4}$ from $5\frac{3}{4}$; (*g*) $3\frac{2}{3}$ from $12\frac{1}{6}$; (*h*) $4\frac{5}{9}$ from 8; (*i*) $23\frac{3}{13}$ from $87\frac{1}{3}$.

6. A roller in a steel mill rolls $112\frac{1}{4}$ tons, $225\frac{5}{8}$ tons, $173\frac{7}{8}$ tons, $185\frac{1}{2}$ tons, and $205\frac{3}{8}$ tons in 5 days. What is the total number of tons rolled?

7. Four automobiles finish a race in the following time: $7\frac{3}{4}$ hr, $8\frac{4}{20}$ hr, $7\frac{4}{5}$ hr, $9\frac{1}{15}$ hr. Find the difference between the winner's time and each of the others.

8. The following entries were found in a weekly summary of the New York Stock Exchange:

Stocks	*Sales, hundreds*	*High*	*Low*	*Close*	*Net Change*
Am. Tel. & Tel.	562	175.2	173.6	175.1	+1.4
duPont.	167	158	149.4	158	+8
Gen. Mot.	437	94.2	90	93	+2.6

The first line, for example, means that American Telephone and Telegraph common stock sold 56,200 shares during the week, had a high selling price of $\$175\frac{2}{8}$ and a low selling price of $\$173\frac{6}{8}$, and ended at a selling price of $\$175\frac{1}{8}$ per share, for a net change of $\$1\frac{4}{8}$ in selling price from the end of the previous week to the end of the current week. For each of the entries, calculate the *range* from low to high selling price during the week.

9. Refer to Problem 8, and calculate the closing selling price for each stock for the previous week.

10. Multiplication of Fractions. It is clear that $1 + 1 + 1 + 1$ equals 4 whole units. It should be equally clear that $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$ equals 4 half units. Since multiplication is essentially a shortened process of addition, the 4 half units would equal $4 \times \frac{1}{2}$, or $\frac{4}{2}$. In general, any fraction may be multiplied by an integer by multiplying the numerator by the integer. Thus, $6 \times \frac{2}{3} = \frac{12}{3}$, $8 \times \frac{7}{5} = \frac{56}{5}$, and so forth.

Suppose we wish to find a fractional part of a fraction. What, for example, is $\frac{1}{3}$ of 9 tenths? The answer, 3 tenths, may be found by taking

$$\frac{1}{3} \times \frac{9}{10} = \frac{9}{30} = \frac{3}{10}.$$

To find $\frac{2}{3}$ of $\frac{9}{10}$, we write

$$\frac{2}{3} \times \frac{9}{10} = \frac{18}{30} = \frac{6}{10} = \frac{3}{5}.$$

In general, to multiply fractions, we multiply numerators for a new numerator and denominators for a new denominator. We may, of course, cancel common factors and thus shorten the work in arriving at the product.

Illustrations:

1. $\dfrac{\cancel{5}}{\cancel{18}} \times \dfrac{\overset{2}{\cancel{36}}}{\underset{9}{\cancel{45}}} = \dfrac{2}{9}$

2. $42\dfrac{2}{3} \times 6\dfrac{1}{4} = \dfrac{\overset{32}{\cancel{128}}}{3} \times \dfrac{25}{\cancel{4}} = \dfrac{800}{3} = 266\dfrac{2}{3}$

3. $\dfrac{40}{50} \times \dfrac{12}{30} \times \dfrac{60}{24} = \dfrac{\overset{4}{\cancel{40}} \times \cancel{12} \times \overset{\cancel{2}}{\cancel{60}}}{\underset{5}{\cancel{50}} \times \cancel{30} \times \underset{\cancel{2}}{\cancel{24}}} = \dfrac{4}{5}$

PROBLEMS

1. Find mentally:
   - (*a*) $\frac{3}{4}$ of $\frac{8}{9}$,
   - (*b*) $\frac{3}{15}$ of $\frac{15}{16}$,
   - (*c*) $\frac{6}{7}$ of $\frac{21}{24}$,
   - (*d*) $\frac{7}{9}$ of $\frac{27}{35}$,
   - (*e*) $\frac{11}{12}$ of $\frac{24}{35}$,
   - (*f*) $\frac{3}{8}$ of $\frac{24}{35}$,
   - (*g*) $\frac{5}{6}$ of $\frac{3}{4}$,
   - (*h*) $\frac{9}{10}$ of $\frac{1}{6}$,

(*i*) $\frac{3}{8} \times \frac{16}{27}$,
(*j*) $\frac{7}{8} \times \frac{40}{42}$,
(*k*) $\frac{75}{100} \times \frac{2}{5}$,
(*l*) $\frac{5}{11} \times \frac{121}{110}$,
(*m*) $\frac{51}{64} \times \frac{8}{17}$,
(*n*) $\frac{5}{8} \times \frac{16}{25}$,
(*o*) $\frac{2}{3} \times \frac{30}{4}$,
(*p*) $\frac{6}{23} \times 46$.

2. Simplify by cancellation:

(*a*) $\dfrac{130 \times 64 \times 19 \times 16}{65 \times 38 \times 8 \times 40}$,

(*b*) $\dfrac{105 \times 58 \times 28 \times 9}{12 \times 54 \times 34 \times 203}$,

(*c*) $\dfrac{3\frac{1}{5} \times 6\frac{2}{3} \times 7\frac{1}{2}}{10\frac{5}{6} \times \frac{2}{13}}$.

3. Find:

(*a*) $\frac{5}{6} \times 10\frac{2}{3}$,
(*b*) $\frac{7}{8} \times 5\frac{1}{3} \times \frac{12}{7}$,
(*c*) $12\frac{1}{5} \times 2\frac{3}{4} \times 6\frac{1}{5}$,
(*d*) $\frac{4}{9} \times 2\frac{1}{2} \times 5\frac{3}{5}$,
(*e*) $8\frac{3}{4} \times \frac{2}{3} \times 5$,
(*f*) $7\frac{5}{6} \times 8 \times 4\frac{4}{5}$,
(*g*) $5\frac{4}{9} \times 3\frac{1}{7} \times 7\frac{7}{8}$,
(*h*) $220 \times 6\frac{4}{5}$,
(*i*) $12 \times 22\frac{1}{32}$,
(*j*) $9 \times \frac{2}{5} \times 12\frac{1}{2}$,
(*k*) $14 \times 1\frac{1}{7} \times \frac{3}{8}$,
(*l*) $16 \times 5 \times 2\frac{1}{40}$,
(*m*) $4\frac{4}{5} \times 3\frac{3}{4}$,
(*n*) $27 \times 5\frac{1}{9} \times 3\frac{2}{3}$,
(*o*) $\frac{1}{27} \times \frac{9}{46} \times \frac{3}{11}$.

4. Refer to Problem 8 of the preceding section and calculate for each entry the profit made by an investor who bought 100 shares of common stock at the closing price of the previous week and sold them at the closing price of the current week.

5. For each entry of Problem 8 of the preceding section calculate the profit realized by an investor who bought 50 shares at the low price for the week and sold them at the high price for the week.

11. Division of Fractions. The instruction $20 \div 4$ means that 20 is to be divided into 4 equal parts, and one of these equal parts can be found by writing $20 \times \frac{1}{4} = 5$. Likewise, $\frac{8}{20} \div 4$ means that $\frac{8}{20}$ is to be separated into 4 equal parts, and one of these equal parts can be found by writing $\frac{8}{20} \times \frac{1}{4} = \frac{2}{20}$. To divide an integer or a fraction by an integer, therefore, it appears that we may invert the divisor and change division to multiplication.

Suppose, now, we wished to find $\frac{8}{20} \div \frac{1}{4}$. Since each unit of the dividend must necessarily contain 4 quarter units, it must follow that the quotient equals $\frac{8}{20} \times 4 = \frac{32}{20}$. Likewise, $\frac{8}{20} \div \frac{4}{5}$ may be written $\frac{8}{20} \times \frac{5}{4} = \frac{40}{80}$, or $\frac{1}{2}$. In general, division by fractions may be accomplished by inverting the divisor and changing division to multiplication.

When mixed numbers are involved, we may change them to improper fractions before starting to divide. Thus,

$$5\frac{4}{9} \div 7\frac{7}{8} = \frac{49}{9} \div \frac{63}{8} = \frac{\overset{7}{\cancel{49}}}{9} \times \frac{8}{\underset{9}{\cancel{63}}} = \frac{56}{81}$$

Illustrations:

1. $\frac{16}{24} \div 8 = \frac{16}{24} \times \frac{1}{8} = \frac{2}{24} = \frac{1}{12}$,
2. $121 \div \frac{11}{2} = 121 \times \frac{2}{11} = 22$,
3. $\frac{5}{8} \div \frac{15}{4} = \frac{5}{8} \times \frac{4}{15} = \frac{1}{6}$,
4. $9\frac{2}{7} \div 13 = \frac{65}{7} \div 13 = \frac{65}{7} \times \frac{1}{13} = \frac{5}{7}$.

Sometimes the division of quantities is indicated in the form of a complex fraction in which the main division is denoted by a heavy line. Thus,

$$\frac{\frac{5}{6} + \frac{1}{3}}{\frac{2}{7}}$$

is a complex fraction in which the numerator, the sum of $\frac{5}{6}$ and $\frac{1}{3}$, is to be divided by $\frac{2}{7}$. The simplification of this expression is as follows:

$$\frac{\frac{5}{6} + \frac{1}{3}}{\frac{2}{7}} = \frac{\frac{5}{6} + \frac{2}{6}}{\frac{2}{7}} = \frac{\frac{7}{6}}{\frac{2}{7}} = \frac{7}{6} \div \frac{2}{7} = \frac{7}{6} \times \frac{7}{2} = \frac{49}{12} = 4\frac{1}{12}.$$

We must keep in mind, when operating with complex expressions, the full intent of the directions given by the form of the expression. A common error arises out of the failure to observe that the heavy line means to divide the entire quantity above it by the entire quantity below it. The following problems will serve to give practice in simplifying a few of the types frequently met:

PROBLEMS

1. Perform the following divisions, simplifying by cancellation when possible:

(*a*) $\frac{2}{3} \div \frac{1}{3}$,
(*b*) $\frac{1}{2} \div \frac{1}{3}$,
(*c*) $\frac{2}{5} \div \frac{3}{4}$,
(*d*) $\frac{3}{4} \div \frac{5}{8}$,
(*e*) $12 \div \frac{1}{4}$,
(*f*) $14 \div 2\frac{1}{3}$,
(*g*) $3\frac{1}{2} \div 5\frac{1}{3}$,
(*h*) $13\frac{2}{3} \div 6\frac{1}{3}$,
(*i*) $15\frac{3}{7} \div 6\frac{4}{9}$,
(*j*) $16\frac{4}{5} \div 21$.

2. Simplify:

(*a*) $\dfrac{\frac{5}{8}}{5}$,

(*b*) $\dfrac{\frac{4}{5}}{2}$,

(*c*) $\dfrac{\frac{3}{4}}{\frac{2}{3}}$,

(*d*) $\dfrac{12}{\frac{3}{5}}$,

(*e*) $\dfrac{6 + \frac{2}{3}}{5}$,

(*f*) $\dfrac{13\frac{1}{5} - 7\frac{2}{5}}{7}$,

(*g*) $\dfrac{2\frac{1}{2} \times 1\frac{3}{5}}{4}$,

(*h*) $\dfrac{5\frac{3}{10} \div 10}{\frac{1}{100}}$,

(*i*) $\dfrac{4(3\frac{1}{2} + 7\frac{3}{4})}{2\frac{1}{2} - \frac{1}{4}}$,

(*j*) $\dfrac{27 \div (5 - 2\frac{3}{4})}{\frac{3}{8}}$,

(*k*) $\dfrac{(\frac{1}{5} \div 5) \times 3\frac{1}{8}}{\frac{1}{8}}$,

(*l*) $(6 \div \frac{3}{4}) - (6 \times \frac{3}{4})$,

(*m*) $\dfrac{\frac{1}{3} + \frac{1}{4} + \frac{1}{5}}{\frac{7}{120} \times 3}$,

(*n*) $\dfrac{\dfrac{(\frac{1}{3} \div \frac{2}{9})}{3} + \frac{3}{2}}{\frac{3}{4} + 3\frac{1}{4}}$,

(*o*) $\dfrac{19\frac{2}{3} - 14\frac{1}{9}}{\frac{100}{27}}$,

(*p*) $\dfrac{6 \times (\frac{3}{4} \div 9)}{\frac{3}{8} - \frac{5}{16}}$,

(*q*) $99 \div \dfrac{\frac{9}{11} + 1\frac{1}{3}}{\frac{4}{9}}$.

3. Simplify:

(*a*) $\dfrac{2\frac{2}{3}}{5\frac{1}{6}}$,

(*b*) $\dfrac{2\frac{1}{2} \times \frac{4}{5}}{\frac{6}{10} \times \frac{3}{7}}$,

(*c*) $\dfrac{3\frac{1}{5} \div 7\frac{1}{9}}{4\frac{1}{3} \div 11\frac{4}{15}}$,

(*d*) $\dfrac{12(5\frac{1}{6} + 9\frac{2}{3})}{7\frac{1}{2} - 4}$,

(*e*) $56 \div \dfrac{4 - \frac{1}{3}}{\frac{4}{7} - \frac{1}{2}}$,

(*f*) $(4 \div \frac{1}{2}) - (\frac{1}{5} \div \frac{4}{5})$,

(*g*) $\dfrac{17\frac{2}{3} - 12\frac{1}{9}}{100}$,

(*h*) $\dfrac{\dfrac{\frac{3}{5} \times \frac{15}{8}}{\frac{1}{9}} + \frac{5}{3}}{\frac{27}{16}}$,

(*i*) $\dfrac{2\frac{1}{3} \div 4\frac{1}{7}}{12\frac{1}{2} \div \frac{5}{3}} + \dfrac{6\frac{1}{8} \times \frac{4}{49}}{\frac{3}{13} \times 26}$,

(*j*) $23 + \dfrac{6\frac{1}{8} - 3\frac{1}{2}}{23 \div 6} - 11\frac{1}{5}$,

(*k*) $\dfrac{12\frac{1}{2} \times 37\frac{1}{2} \times 66\frac{2}{3}}{33\frac{1}{3} \times 6\frac{1}{4} \times 75}$.

4. What number multiplied by $\frac{3}{5}$ will give 10 for a product?

5. What part of 3 is $\frac{2}{3}$ of 2?

6. What part of $1\frac{1}{4}$ ft is $3\frac{1}{3}$ in.?

7. A student read 96 pages of a 432-page book. What part of the book remained to be read?

8. What part of a day is 13 hr 30 min?

9. A floor tile $\frac{9}{16}$ sq ft in area sells for 9¢. What is the price per square foot?

10. Carpeting 27 in. wide sells for $6 a yard. What is the price per square foot?

12. Decimals. The divisions of an integral unit into 10, 100, 1,000 (and so forth) equal parts are called *decimal* divisions. These divisions may be expressed, in words, as seven-tenths, twelve-hundredths, and so forth, or they may be stated, as common fractions, as $\frac{7}{10}$, $\frac{12}{100}$, and so forth, or they may be written as *decimal fractions*, or *decimals*, in which a period, or decimal point, is employed. The position of a digit to the right of the decimal point determines whether it represents tenths, hundredths, and so forth; its position to the left of the point determines whether it represents ones, tens, hundreds, and so forth. Thus, 24.375 means that there are 2 tens, 4 ones, 3 tenths, 7 hundredths, and 5 thousandths, and the number is read "24 and 375 thousandths" as an abbreviated equivalent. In reading a decimal number, the word "and" should be said only when the decimal point is reached. Thus, 4,862.628 is read "four thousand eight hundred sixty-two *and* six hundred twenty-eight thousandths." We have a familiar example of decimal quantities in the writing of dollars and cents; for example, $24.63 means "twenty-four and sixty-three hundredths dollars."

Problems

1. Read: (*a*) 0.25; (*b*) 0.06; (*c*) 0.002; (*d*) 0.225; (*e*) 0.203; (*f*) 0.0005; (*g*) 0.0092; (*h*) 0.32578; (*i*) 0.243675; (*j*) 47.6; (*k*) 295.65; (*l*) 1,211.005; (*m*) 1,001.1001; (*n*) 8,000.008; (*o*) 700.07; (*p*) 0.00875.

2. Write as fractional expressions: (*a*) 0.25; (*b*) 0.9; (*c*) 0.045; (*d*) 0.0025; (*e*) $0.37\frac{1}{2}$; (*f*) $0.06\frac{1}{4}$; (*g*) $0.12\frac{1}{2}$; (*h*) $0.66\frac{2}{3}$; (*i*) $0.14\frac{2}{7}$; (*j*) $0.62\frac{1}{2}$; (*k*) $0.33\frac{1}{3}$; (*l*) $0.83\frac{1}{3}$.

13. Addition and Subtraction of Decimals. In addition and subtraction it is necessary to keep in the same column all digits having the same place value with respect to the decimal point. The decimal points should, therefore, be kept in a column.

Problems

1. Add as indicated, and check results:

$$\begin{array}{rcrcrcrcl}
0.37 & + & 84.24 & + & 6.7 & + & 0.807 & = & \\
7.56 & + & 8.032 & + & 0.405 & + & 1.55 & = & \\
16.257 & + & 115.25 & + & 75. & + & 10.003 & = & \\
\underline{50.316} & + & \underline{0.462} & + & \underline{2.35} & + & \underline{8.294} & = & \underline{\qquad}
\end{array}$$

2. Change first to decimals, and then add:

(a) $5\frac{1}{2} + 66\frac{1}{4} + \frac{1}{8}$,
(b) $\frac{27}{2} + \frac{5}{9} + 164\frac{5}{8}$,
(c) $\$1\frac{3}{4} + \$2\frac{1}{4} + \$6\frac{1}{2} + \$15\frac{7}{8}$,
(d) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$.

3. Complete the following statement of profit:

Gross Profit		$97,024.80
Expenses		
Selling expenses	$42,480.37	
Administrative expenses	21,500.00	
Overhead expenses	2,376.72	
Other expenses	1,294.05	
Total Expenses	$	
Net Profit		$______

4. Balance the following cash account:

Cash Account

Date	*Receipts*		*Date*	*Expenses*	
Mar. 1	Cash on hand	$37.55	Mar. 1	Streetcar tickets	$1.10
2	Part-time work	14.00	5	College play	.75
3	Allowance	4.00	6	Cigarettes	2.30
			7	Bond paper	.60
			8	Balance	
Mar. 8	Cash on hand	$			

5. Complete the balance column of the following bank statement:

Checks Paid			*Deposits*	*Balance*	*Date*
Balance Brought Forward July 1				$457.39	1
$15.35	$ 2.45	$22.74			
				398.88	3
2.54	5.48	9.95			
			$197.50	596.38	4
					14
54.60	23.71				
					20
32.90					
					23
			23.84		
					25
12.53					
					29
8.43	20.05				
					31
2.55					

6. Complete the following statement of sales:

Gross Sales		$87,264.45
Discounts	$ 1,375.92	
Returns	13,458.94	
Freight	237.38	
Deductions	$	
Net Sales		$________

14. Multiplication of Decimals. When numbers involving decimals are multiplied together, the number of decimal places in the product equals the sum of the number of decimal places in the multiplier and multiplicand. Thus $0.004 \times 0.02 = 0.00008$, a product having three plus two, or five, decimal places. In all other respects the multiplication is the same as for whole numbers.

In most business problems decimal products are approximated to two or more decimal places. Usually, where money is involved, the product is rounded off to two decimals—that is, to the nearest cent. It should be pointed out that the rounding off should take place in the product *after* the numbers are multiplied, not in the individual factors before they are multiplied. Otherwise, the approximation may be too far off to be of practical use. The process of rounding off will be discussed in Section 17.

PROBLEMS

1. Perform the following multiplications:

(*a*) 3×0.003,
(*b*) 0.03×0.3,
(*c*) 4×0.12,
(*d*) 2.4×10,
(*e*) 13×2.15,
(*f*) 4.7×23.45,
(*g*) 150×0.25,
(*h*) 5.25×67,
(*i*) 25.875×2.5.

2. Find the cost of:

(*a*) 15 lb sugar at $0.13,
(*b*) 7 doz oranges at $0.35,
(*c*) 8 lb coffee at $0.88,
(*d*) 14 yd cloth at $$0.67\frac{1}{2}$,
(*e*) $6\frac{1}{4}$ lb meat at $$0.62\frac{1}{2}$,
(*f*) $16\frac{2}{3}$ lb butter at $0.96,
(*g*) $2\frac{1}{2}$ gal cider at $$0.33\frac{1}{3}$,
(*h*) 5 cans cleaner at $$0.08\frac{1}{3}$.

3. Find the amount of each of the following sales, and calculate the totals:

(*a*) 48 yd at $0.12½ =
124 yd at 0.37½ =
78 yd at 0.62½ =
84 yd at 0.87½ =
132 yd at 0.06¼ =
Total $_____

(*b*) 12 cases corn at $5.08 =
25 cases beans at 6.87 =
7 cases beets at 4.20 =
38 cases peaches at 7.12½ =
12 cases peas at 6.15 =
Total $_____

4. Find the total pay roll:

Number of Employees	*Hours Worked per Employee*	*Rate per Hour*	*Amount*
67	44	$2.40	$_____
108	42½	2.64	_____
124	46	2.32	_____
135	45	2.20	_____
192	43½	2.80	_____
207	43	2.70	_____
		Total	$_____

5. Find the total production cost:

Item	*Number Produced*	*Cost Each*	*Amount*
A	224	$11.24	$_____
B	68	7.45	_____
C	309	2.25	_____
D	152	19.20	_____
E	69	45.96	_____
		Total	$_____

15. Division of Decimals. When a decimal is divided by a whole number the decimal point in the quotient is placed directly above the decimal point in the dividend. In case the divisor is not a whole number, it can be made a whole number by multiplying by 10, 100, 1,000, and so forth, as needed. But then we must multiply the dividend by the same multiple of 10. All this amounts to moving the decimal point to the right in both divisor and dividend the same number of places before dividing. The decimal point in the quotient is then placed directly above the decimal point in the dividend; and division proceeds as in the case of whole numbers. The following illustrations will serve to review the process:

Illustrations:

1. Find the quotient of 42.578 divided by 18.24.

$18.24\overline{)42.578}$ (Problem set up)

$1824\overline{)4257.8}$ (Decimal point moved two places to the right in both divisor and dividend)

$$\begin{array}{r} . \\ 1824\overline{)4257.8} \end{array}$$ (Decimal point of quotient placed in position)

$$\begin{array}{r}
2.33 \\
1824\overline{)4257.80} \\
\underline{3648} \\
6098 \\
\underline{5472} \\
6260 \\
\underline{5472} \\
788
\end{array}$$ (Quotient carried out to two decimal places)

2. Place the decimal point of the quotient in position for each of the following problems:

(*a*) $24.352\overline{)1.046}$

$24352\overline{)1046.}$ (Decimal point moved three places to the right)

$$\begin{array}{r} . \\ 24352\overline{)1046.} \end{array}$$ (Decimal point placed in quotient)

(*b*) $6.459\overline{)12.}$

$6459\overline{)12000.}$ (Decimal point moved three places to the right, with zeros filling the vacant spaces)

$$\begin{array}{r} . \\ 6459\overline{)12000.} \end{array}$$ (Decimal point placed in position)

PROBLEMS

1. Find the following quotients to the number of decimal places indicated in the parentheses to the right:

(*a*) 587 ÷ 23.4 (2),
(*b*) 36.7 ÷ 58.36 (2),
(*c*) 2.409 ÷ 0.045 (3),
(*d*) 874.37 ÷ 968 (2),
(*e*) 0.0054 ÷ 1.8 (3),
(*f*) 5,306 ÷ 7,835 (3),

2. Perform the following long-division problems, carrying out the quotient to the number of decimal places indicated in parentheses:

(*a*) 723 ÷ 64 (2),
(*b*) 1,692 ÷ 75 (3),
(*c*) 43.65 ÷ 27.2 (2),
(*d*) 1.06 ÷ 2.1 (3),
(*e*) 100.67 ÷ 31.24 (2),
(*f*) 69,084 ÷ 6,784 (1),
(*g*) 0.0768 ÷ 3.006 (2),
(*h*) 0.0054 ÷ 0.086 (4),
(*i*) 3.14159 ÷ 4.6 (5),
(*j*) 27.001 ÷ 0.00013 (1).

3. If 12 government bonds cost \$225, find the cost of each bond.

4. How many articles can be bought for:

(*a*) \$84 at \$0.12 each?
(*b*) \$125.50 at \$2.50 each?
(*c*) \$16 at \$0.06¼ each?
(*d*) \$725.40 at \$22.50 each?
(*e*) \$3,750.50 at \$108 each?
(*f*) \$4.28 at \$0.02½ each?

5. A certain salesman received a bonus every time he sold \$75.50 worth of goods. How much was each bonus if he received \$82.50 for selling a total of \$1,132.50 worth of goods?

6. An automobile travels 209 miles in 5 hr 30 min. What is the average rate of travel?

7. A hotelkeeper purchased:

\$49.60 worth of rice at 15½¢ per pound,
\$350 worth of sugar at 9¢ per pound,
\$52 worth of soap at the rate of 2 cakes for 25¢,
\$107.25 worth of coffee at 71½¢ per pound.

Find the quantity of each item purchased.

16. Approximate Numbers and Significant Figures. In many situations where observations or measurements are made, the numbers representing the collected data have been determined by some process of measuring. For example, it may be recorded that a metal bar has a length of 16.2 cm, a weight of 2.3 lb, and so forth. Such numbers, unlike *exact* numbers resulting from counting objects, are *approximate* numbers since they are essentially comparisons of their magnitudes with standards or instruments that are not perfect. Since the student will be dealing with approximate numbers in many business and industrial problems, it is important that he know how to indicate the degree of the error of the measurement and how to use approximate numbers correctly in computations involving them.

The degree of error involved in an approximate number is usually expressed by the form in which it is written. For example, by writing 16.2 cm we mean that the "true value" of the measurement is presumed to lie between 16.15 and 16.25 cm, with 16.2 cm taken to be the most probable value. It is seen, therefore, that the digit 2 in this approximate number is inaccurate; but it carries significance along with the other recorded digits, since it serves to complete the accepted value of the measurement. Thus the approximate number 16.2 is said to be a number of three *significant figures* (1, 6, and 2). As another illustration, let us examine the approximate weight 0.048 lb. The true weight is presumed to lie between 0.0475 and 0.0485 lb, with 0.048 believed to be the most probable value. We say that this number has two significant figures (4 and 8)—the first nonzero digit, counting from the left, and the inaccurate figure itself. For *decimal fractions*, the rule for counting the *number of significant figures* is to *begin at the first nonzero digit at the left and count every digit to the right, including the last, or inaccurate, digit.*

This rule is to be followed regardless of whether the last digit is zero. Thus the number of significant figures in the approximate number 0.07080 is four—the count obtained by starting at the 7 and proceeding to the right. Notice that the position of the decimal point has nothing to do with the number of significant digits; that is, 0.007080, 7.080, 70.80, etc., would all be numbers of four significant figures. The following table will serve to give further clarification of this rule for counting significant figures in decimal fractions:

Approximate Number	*Number of Significant Figures*
0.00071	2
0.06000	4
56.5008	6
289.37	5
16.250	5

In designating the number of significant figures in *integers*, consecutive zeros on the far right may or may not be regarded as significant, depending on the accuracy of the measurements. Thus, the measurement 60 ft could mean that the true value is presumed to lie between 59.5 and 60.5 ft; or it could mean that the probable true value lies between 50 and 70 ft. In the first interpretation, accuracy to the nearest foot is implied, and we say that both digits, the 6 and the 0, are significant. In the second interpretation, accuracy to the nearest 10 ft is indicated; and we would say that only the first digit, the 6, is significant. Hence, the number of significant figures in an integer ending in zeros can be stated only when the accuracy of the measurement is known.

17. Rounding Off Numbers. When a number contains more digits than we care to use, we may approximate it by a number with fewer digits by the process of *rounding off*. For example, to the nearest cent we would round off sums of money as follows:

$24.278, rounded off as $24.28,
$15.673, rounded off as $15.67,
$92.135, rounded off as $92.14,
$63.245, rounded off as $63.24.

The usual procedure in rounding off may be summarized as follows:

1. *If the part of the number to be discarded amounts to more than* 5 *units in the first discarded place, increase the last digit retained by* 1 *unit.*

2. *If the part discarded amounts to less than* 5 *units in the first discarded place, do not change the last digit retained.*

3. *If the part discarded amounts to exactly* 5 *units in the first discarded place, leave the last digit unchanged if it is even but increase it by* 1 *unit if it is odd.*

Illustrations:

(1) 18.3762 = 18.38 to two decimal places,
4.6501 = 4.7 to one decimal place,
0.00018 = 0.0002 to four decimal places.

(2) 14.26398 = 14.26 to two decimal places,
216.9499 = 216.9 to one decimal place,
0.0617 = 0.06 to two decimal places.
(3) 65.4500 = 65.4 to one decimal place,
8.375 = 8.38 to two decimal places,
0.00155 = 0.0016 to four decimal places.

The process of rounding off is used in many situations involving approximate numbers. Important applications may be found in stating dollar amounts to the nearest cent, in printing values found in tables of interest, annuities, and logarithms, and so forth. It should be pointed out, however, that in many business practices *any* fractional part of a cent is counted as an extra cent; in such cases the rules for rounding off do not apply.

18. Operations with Approximate Numbers. In performing operations with approximate numbers, both *accuracy* and *precision* must be taken into account. Accuracy refers to the number of significant figures in the number, whereas precision refers to the number of decimal places or units retained. Thus, the approximate numbers 6.4 and 0.68 are equally accurate in that each has two significant figures; the numbers 6.4 and 0.7 would have the same precision, since each is carried to the same number of decimal places. It would be rather difficult to state a complete set of rules for expressing results in every series of computations with approximate numbers, but *working rules* can be stated that will tend to eliminate unnecessary work and at the same time lead to results compatible with the accuracy and the precision of the numbers involved.

To illustrate what happens when two approximate numbers are multiplied, let us consider the area of a rectangular mat measured as 6.1 in. long and 4.7 in. wide. From the meaning of significant figures, we know that each of these numbers is inaccurate in the last digit. Consequently, when they are multiplied to find the area, there must be a question as to the number of digits to be retained in the product. One might say that the area is

$$6.1 \text{ in.} \times 4.7 \text{ in.} = 28.67 \text{ sq in.}$$

But would this display of accuracy be justified by our measurements? Probably not. Since 6.1 indicates that the true length is somewhere between 6.05 and 6.15 in. and since the corresponding interval of confidence for the width measurement is from 4.65 in. to 4.75 in., allowance must be made for a suitable range within which the true area may be presumed to lie. In fact, we can say only that the area of the mat must be less than

$$6.15 \text{ in.} \times 4.75 \text{ in.} = 29.2125 \text{ sq in.}$$

and greater than

$$6.05 \text{ in.} \times 4.65 \text{ in.} = 28.1325 \text{ sq in.}$$

To record the area as 28.67, or even as 28.7, sq in. would therefore mean stating more accuracy for the area than is justified. Without further information, persons trained in the use of approximate numbers would record the area as 29 sq in. This result is the same as that obtained by rounding off the product 28.67 to two significant figures. Since a similar situation is encountered in the division of two approximate numbers, the following working rule is suggested:

The result of a multiplication or a division of approximate numbers should not be given with more significant figures than are possessed by that one of the numbers having the fewest significant figures.

Illustrations:

(*a*) $5.64 \times 2.7 = 15.$

$$\begin{array}{r} 5.64 \\ \underline{2.7} \\ 3\,948 \\ \underline{11\,28} \\ 15.228 \end{array}$$

(*b*) $8.7395 \div 3.24 = 2.70.$

$$\begin{array}{r} 2.697 \\ 3.24\overline{)8.73950} \\ \underline{6\,48} \\ 2\,259 \\ \underline{1\,944} \\ 3155 \\ \underline{2916} \\ 2390 \\ \underline{2268} \end{array}$$

To state the result of the addition or the subtraction of approximate numbers, one must take into account the precision of the numbers involved. The working rule may be stated as follows:

The result of an addition or a subtraction of approximate numbers should not be given with more precision than is found in that number having the greatest error in measurement.

Ordinarily, in working with decimals this rule can be interpreted to mean: *Do not retain more decimal places in the result than occur in that number having the fewest decimal places.* If the student prefers to do so, he may retain all given digits in the intermediate work of an addition or a subtraction problem and then round off the final result according to the preceding rule. An alternative procedure, which may save needless work, is to round off numbers of the data to one more place than is to be retained in the answer and then round off the answer according to the rule.

Illustrations:

(*a*) Add:

$$\begin{array}{rlcrl}
\$457.923 & & & \$457.923 & \\
23.17 & & & 23.17 & \\
2.3028 & & \text{or} & 2.303 & \\
\underline{3{,}945.24823} & & & \underline{3{,}945.248} & \\
\$4{,}428.64403 & = \$4{,}428.64 & & \$4{,}428.644 & = \$4{,}428.64
\end{array}$$

(*b*) Subtract:

$$\begin{array}{rlcrl}
\$2{,}764.8457 & & & \$2{,}764.846 & \\
\underline{251.91} & & \text{or} & \underline{251.91} & \\
\$2{,}512.9357 & = \$2{,}512.94 & & \$2{,}512.936 & = \$2{,}512.94
\end{array}$$

The working rules that have been given for operations with approximate numbers will not necessarily give the best results in every situation. Any effort to give an all-inclusive discussion would scarcely be worth while because of the complexity of the sequence of operations in some problems. In extended mixed operations, it is good practice to retain in all calculations at least one more digit than will be retained in the final rounded-off result.

PROBLEMS

1. Express 2 ft 8 in. as feet correct to 2 decimal places. How many significant figures are in the answer?

2. Round off the following amounts correct to the nearest cent: (*a*) \$674.348, (*b*) \$37.283, (*c*) \$5,864.039, (*d*) \$35.235.

3. Find the value of the product of 6.2368 × 86.243, and then round off the answer correct to two decimal places.

4. Find the following quotients to the nearest third significant decimal place:

(*a*) $567 \div 34$,
(*b*) $2.9674 \div 784$,
(*c*) $0.0457 \div 0.89563$,
(*d*) $2{,}365 \div 8{,}643$.

5. Find the value of 1.06 × 1.06 × 1.06 × 1.06 correct to two decimal places.

6. What difference results from computing the product 1.036 × \$1.036 by rounding off to two decimal places before and after multiplication and by rounding off to two decimal places only after the multiplication has been performed?

7. State the number of significant figures in the following measurements: (*a*) 27, (*b*) 396, (*c*) 26.5, (*d*) 0.5, (*e*) 0.05, (*f*) 0.005, (*g*) 2.70, (*h*) 2.700, (*i*) 4.0004, (*j*) 127,000, (*k*) 0.127, (*l*) 10.010, (*m*) 0.01101, (*n*) 26.55, (*o*) 1.1, (*p*) 100, (*q*) 0.00623, (*r*) 1.673, (*s*) 0.1768253, (*t*) 384.56, (*u*) 2,687.006, (*v*) 2,397,600.

In the following problems assume all numbers to be approximate, and perform the indicated operations, retaining only as many significant figures in the result as are justifiable:

8. (*a*) 11.2×0.65,
(*b*) 103.2×10.0,
(*c*) $0.532 \times 0.101 \times 2.01$,
(*d*) 0.002×6.31,
(*e*) $101.0 \times 1{,}111.1$,
(*f*) 1.002×10.20.

9. (*a*) 14.2/1.0,
(*b*) 1.234/6.1,
(*c*) 0.0043/1.21,
(*d*) 5,623/5.6,
(*e*) 22.4/1.1,
(*f*) 1.3/1.12.

10. (*a*)	23.27	(*b*)	121.1	(*c*)	9.0
	13.541		12.12		0.12
	22.76		45.501		62.45
	106.892		22.23		12.45

11. Subtract:

(a) $\begin{array}{r} 125.62 \\ \underline{14.21} \end{array}$ (b) $\begin{array}{r} 165.865 \\ \underline{134.20} \end{array}$ (c) $\begin{array}{r} 0.00635 \\ \underline{0.0041} \end{array}$

19. Checking Computations. Various methods of checking the results of the operations of addition, subtraction, multiplication, and division have been advocated. No check, of course, will take the place of careful, accurate work; but some check should always be applied to test the correctness of results obtained.

(a) **Addition:** Addition may be checked in several ways, such as addition in a reverse order or combining numbers by a different arrangement. A method which is easy to apply and which is especially useful where long columns are involved is known as the method of *casting out nines.* The method consists essentially in determining whether the sum of the digits in each number in excess of some multiple of nine is in aggregate equal to the corresponding excess in the answer.

Illustration: Check the sum of 45.43, 8.85, 247.08, 7.33, and 0.17 by casting out nines.

$$\begin{array}{r} 45.43 \\ 8.85 \\ 247.08 \\ 7.33 \\ \underline{0.17} \\ 308.86 \end{array} \qquad \begin{array}{l} \textit{Check:} \\ \begin{aligned} 4 + 5 + 4 + 3 &= 16 = 1 \times 9 + 7, \\ 8 + 8 + 5 &= 21 = 2 \times 9 + 3, \\ 2 + 4 + 7 + 0 + 8 &= 21 = 2 \times 9 + 3, \\ 7 + 3 + 3 &= 13 = 1 \times 9 + 4, \\ 1 + 7 &= 8 = 0 \times 9 + 8, \\ 3 + 0 + 8 + 8 + 6 &= 25 = 2 \times 9 + 7, \\ 7 + 3 + 3 + 4 + 8 &= 25 = 2 \times 9 + 7. \end{aligned} \end{array}$$

The sum of the digits of 45.43 is seen to be 7 in excess of 9; for the number 8.85, the sum of the digits is seen to be 3 in excess of 18, and so forth. When all these excess numbers are totaled, the sum is 25, which is 7 more than 18. This corresponds to the excess obtained when the digits of the answer (308.86) are summed. Since the two final results are the same, the addition checks. It should be pointed out, however, that although the equality of the two excesses is necessary for the sum to be correct, it does not guarantee the correctness of the sum. For example, an incorrect sum, say 308.68, would also have checked.

PROBLEMS

1. Check the sum of 467.32, 78.56, 1.25, and 85.45 by casting out nines.

2. Check the sums obtained in Problem 1, Section 1, by casting out nines.

(b) Subtraction: The best check for subtraction is to add the remainder to the subtrahend to obtain the minuend.

Illustration: Check the difference between 467,564 and 279,632.

Check:

$$\begin{array}{r} 467{,}564 \\ \underline{279{,}632} \\ 187{,}932 \end{array} \qquad\qquad \begin{array}{r} 187{,}932 \\ \underline{279{,}632} \\ 467{,}564 \end{array}$$

PROBLEMS

Perform the following subtractions, and check results:

$$\text{(1)}\ \begin{array}{r} 6{,}985 \\ \underline{3{,}978} \end{array} \qquad \text{(2)}\ \begin{array}{r} 985.47 \\ \underline{579.35} \end{array} \qquad \text{(3)}\ \begin{array}{r} 9{,}854.68 \\ \underline{4{,}782.07} \end{array} \qquad \text{(4)}\ \begin{array}{r} 86{,}497 \\ \underline{29{,}040} \end{array}$$

(c) Multiplication: By interchanging multiplier and multiplicand and multiplying again, the results of multiplication may be checked.

Illustration: Check the product of 483 × 253.

Check:

$$\begin{array}{r} 483 \\ \underline{253} \\ 1449 \\ 2415 \\ 966 \\ \hline 122199 \end{array} \qquad\qquad \begin{array}{r} 253 \\ \underline{483} \\ 759 \\ 2024 \\ 1012 \\ \hline 122199 \end{array}$$

Multiplication may also be checked by casting out nines. An example will serve to make the method clear.

$$\begin{array}{r l}
483 & 4+8+3 = 1\times 9+6,\\
\underline{253} & 2+5+3 = 1\times 9+1,\\
1449 & \\
2415 & \qquad\qquad 1\times 6 = 6,\\
\underline{966} & \\
122199 & 1+2+2+1+9+9 = 2\times 9+6.
\end{array}$$

1 × 6 is the product formed by multiplying the excesses in the sums of the digits of the multiplier 253 and the multiplicand 483. Since this product has an excess of 6 and the answer likewise has an excess of 6, the multiplication checks.

Problems

Perform and check the following multiplications:

$$\begin{array}{r r r r}
(1)\ 765.48 & (2)\ 56.458 & (3)\ 76985 & (4)\ 356.973\\
\underline{479.80} & \underline{22.008} & \underline{3794} & \underline{32.065}
\end{array}$$

(d) Division: To check division, the most common procedure is to multiply the quotient by the divisor to obtain the dividend. If there is a remainder involved, it must be added to the product of the quotient and the divisor.

Illustration: Divide 5,847 by 373, and check result.

$$\begin{array}{r}
15\\
373\overline{)5847}\\
\underline{373}\\
2117\\
\underline{1865}\\
252
\end{array}
\qquad
\begin{array}{r}
\textit{Check:}\\
373\\
\times\ \underline{15}\\
1865\\
\underline{373}\\
5595\\
+\ \underline{252}\\
5847
\end{array}$$

Problems

Perform the following divisions, and check the results:

(1) $47.58\overline{)6{,}284.67}$, (2) $9.574\overline{)67.3456}$, (3) $47\overline{)9.0302}$, (4) $0.032\overline{)0.0256}$.

Exercises

1. Perform the indicated operations:

(*a*) $\frac{3}{4} + \frac{1}{2} + \frac{1}{8}$

(*b*) $2\frac{1}{6} + 1\frac{1}{3} + 4\frac{5}{9}$,

(*c*) $\frac{7}{12} - \frac{1}{6}$,

(*d*) $3\frac{3}{4} \times 2\frac{1}{2} \times \frac{3}{5}$,

(*e*) $\frac{1}{2} \times \frac{2}{7}$,

(*f*) $5\frac{2}{3} - \frac{3}{7}$,

(*g*) $3\frac{5}{8} \times 2\frac{1}{2} \times \frac{3}{5}$,

(*h*) $\frac{2}{3} \div \frac{20}{9}$,

(*i*) $\frac{1}{2} \times \frac{3}{4} \div \frac{5}{8}$,

(*j*) $3\frac{3}{5} \times 2\frac{1}{9} \div 19$.

2. Add:

(*a*) $\frac{1}{3} + \frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \frac{1}{6}$,

(*b*) $\frac{10}{15} + \frac{16}{18} + \frac{24}{26} + \frac{35}{45} + \frac{42}{96}$,

(*c*) $2\frac{1}{2} + 3\frac{1}{3} + 4\frac{1}{4} + 5\frac{1}{5}$,

(*d*) $10\frac{4}{7} + 14\frac{9}{11} + 132\frac{18}{77}$,

(*e*) $\frac{3}{8} + \frac{2}{7} + \frac{2}{5} + \frac{5}{9}$.

3. Subtract as indicated:

(*a*) $\frac{3}{8} - \frac{1}{4}$,

(*b*) $5\frac{2}{3} - 2\frac{1}{6}$,

(*c*) $27\frac{5}{13} - 19\frac{25}{39}$,

(*d*) $7\frac{3}{7} - 6\frac{4}{7}$,

(*e*) The sum of $\frac{2}{3}$ and $\frac{5}{9}$ from the sum of $\frac{1}{6}$ and 4,

(*f*) The sum of $\frac{2}{7}$ and $\frac{1}{3}$ from the difference of $2\frac{5}{42}$ minus $\frac{15}{21}$,

(*g*) $124\frac{1}{5} - 92\frac{3}{4}$.

4. Multiply:

(*a*) $\frac{11}{14} \times 7$,

(*b*) $\frac{3}{7} \times \frac{5}{9}$,

(*c*) $3\frac{3}{4} \times 7$,

(*d*) $5\frac{2}{3} \times 4\frac{1}{2}$,

(*e*) $11\frac{4}{15} \times 23\frac{1}{169}$,

(*f*) $6\frac{7}{8} \times 5\frac{3}{11} \times \frac{2}{29}$.

Divide:

(*g*) $\frac{4}{5}$ by $\frac{5}{7}$,

(*h*) $3\frac{5}{6}$ by 5

(*i*) $1\frac{4}{5}$ by $3\frac{5}{7}$,

(*j*) $\frac{9}{11}$ by $\frac{13}{44}$,

(*k*) $27\frac{1}{2}$ by $\frac{5}{4}$,

(*l*) 128 by $6\frac{1}{2}$.

5. If outer and inner dimensions of a metal tube are $5\frac{1}{2}$ in. and $3\frac{13}{16}$ in., respectively, how thick is the tube?

6. The production of the Lofty Aircraft Company is 70 units per hour. If the working day is 8 hr, find the total production in a month of 24 working days.

7. The Mighty Machine Shop employs 632 men. The weekly pay roll is $45,050. What is the average weekly wage of each employee?

8. Change the following speeds in miles per hour to feet per second: (*a*) 45 mph, (*b*) 185 mph, (*c*) 320 mph, (*d*) 2 mph.

9. Change the following speeds in feet per second to miles per hour: (*a*) 25 ft per sec, (*b*) 88 ft per sec, (*c*) 160 ft per sec.

10. Divide and express the answers correct to three decimal places: (*a*) 4.372 by 264, (*b*) 56.5 by 0.046, (*c*) 0.00468 by 0.2842, (*d*) 0.0036 by 0.0002.

11. In testing certain manufactured bolts, each bolt is allowed a tolerance of $\frac{1}{50}$ of its length. One specific bolt supposed to be 6 in. long is found to be $6\frac{1}{8}$ in. long. Is it acceptable? What are the shortest and the longest "6-inch" bolts that would be acceptable?

12. The Lowcut Company manufactures two grades of shoes, the first grade at a cost of $12 a pair, and the second grade at a cost of $5. If 475 pairs of each kind are produced each day, find the total production cost.

13. How many pieces each 3.52 in. long can be cut from a length of metal stock 40.26 in. long? How much is left if 0.03 in. is wasted for each cut?

14. Find the prime factors of 1,728; 832; 1,125; 1,722; 3,928; 8,625.

15. Find the GCD of the following: (*a*) 172 and 1,290; (*b*) 56 and 72; (*c*) 9,757 and 100,100; (*d*) 9, 27, 33, and 14; (*e*) 66, 132, 220, 330, and 462.

16. Find the LCM of the following: (*a*) 8, 16, 24, 48; (*b*) 72, 36, 108, and 96; (*c*) 17, 34, 51, and 54; (*d*) 1, 6, 12, 18, and 22.

17. A bar of metal 17.6 in. long weighs 4.65 lb. What is the weight per inch of length correct to one decimal place?

18. Find the cost of: (*a*) 15 doz eggs at $0.75 per dozen, (*b*) $16\frac{2}{3}$ lb rice at 0.11\frac{1}{9}$ per pound, (*c*) $87\frac{1}{2}$ yd of material at 90¢ a yard, (*d*) 72 hand-painted neckties at $2.25 each.

19. At $215.50 per front foot, how much will a lot with a frontage of 90 ft cost?

20. If 36 boxes of canned goods cost $243, how much will 65 boxes cost?

21. A real estate agent sold 256 acres of land at $140 an acre and 192 acres at $180 an acre. He invested the money in city lots at $1,100 each. How many lots did he buy?

22. A town has a population of 7,200. If the average amount of water used by each person is $8\frac{3}{4}$ gal per day, find the number of gallons of water used in a 30-day period.

23. A recreation room is $72\frac{2}{3}$ ft long and $25\frac{1}{4}$ ft wide. At $12\frac{1}{2}$¢ per foot, how much will it cost to put a molding around the room?

24. An heir gets $\frac{8}{9}$ of an estate and invests $\frac{5}{8}$ of his share and still has $3,200 left. What is the value of the estate?

25. In closing out his goods, a merchant sold $0.47\frac{1}{2}$ of the stock the first month, 0.25 the second month, and the remainder, worth $16,500, the third month. What was the value of his stock of goods?

26. Three oil tankers leave New York for Jacksonville on the same day. It takes the first tanker 6 days, the second 7 days, and the third 9 days to make a round trip. How many days will pass before all three ships leave New York again on the same day?

27. Mr. Adams owns $\frac{5}{6}$ of the total stock of a foundry and sells $\frac{21}{25}$ of his share. What part of the total stock does he still own?

28. R. E. Trench owns $6,000 worth of stock in a corporation whose capital stock is $60,000. If he sells 0.16 of his shares, what is the value of the stock he then owns? What part of the corporation's total stock did he sell?

29. A contractor wants to hire 7 men for $10\frac{1}{2}$ days. He has $929.04 to pay them. What daily wage can he promise them?

30. A beverage-dispensing machine holds $5\frac{1}{4}$ gal. Glasses of the beverage are sold for 5¢. The beverage costs the dealer $0.32 per gallon. After 32 glasses are sold, $\frac{12}{21}$ of the original amount remains. How much profit does the dealer make on a filling of the dispenser?

31. Three pieces of tape, $12\frac{1}{4}$ in., $7\frac{2}{3}$ in., and $8\frac{1}{6}$ in. long, respectively, are spliced together. Each splice requires $1\frac{1}{4}$ in. from the end of each spliced tape. What will be the length of the tape when spliced?

32. S. A. Smith gets $15\frac{3}{7}$ miles per gallon of gasoline with his automobile. How many gallons of gasoline will be required for a trip of 350 miles?

33. A 2-qt, a 3-qt, and an 8-qt measure are available at a filling station for filling radiators with antifreeze. What is the smallest-sized radiator that could be filled exactly by using any one of the three measures if each measure is to be full, and emptied each time?

34. What is the longest piece of rope that may be used in measuring lengths of 180 yd, 108 yd, 396 yd, and 144 yd?

35. A fifth of a gallon of a certain beverage cost $3.10. How much does it cost per pint?

36. A drinking glass holds half a pint. How many glasses of liquid could be obtained from $4\frac{5}{16}$ gal?

37. A man builds a house costing $21,000 and immediately insures it for $\frac{4}{7}$ of its value. It burns down, and the insurance company allows him $\frac{4}{9}$ of the original cost. What fractional part of his insurance did he collect?

38. If crayons sell for 29¢ per dozen, what should one pay for 4 crayons? For 5?

39. On Jan. 1, 1944, there were 6 licensed commercial television stations in the United States. On Jan. 1, 1953, there were 96. What fractional part of the former number is the number representing the increase?

40. In 1950, there was a total of 5,382,000 farms in the United States. Of these 66,500 were in South Dakota. What decimal part of the total did South Dakota possess? (Answer correct to 2 significant figures.)

41. Refer to Problem 40, and determine that decimal part of the farms possessed by Pennsylvania, in which state there were 146,900 farms.

42. How many articles can be bought for (*a*) $27 at $$0.33\frac{1}{3}$ each, (*b*) $6 at $$0.06\frac{1}{4}$ each, (*c*) $6.50 at $$0.08\frac{1}{3}$ each, (*d*) $$37\frac{1}{2}$ at $0.75 each?

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43. If copper wire weighs 38.76 lb per 1,000 lin ft, what will be the weight of a line connecting two cities 260 miles apart?

44. From an ice-cream can which was $\frac{7}{8}$ full, 48 dishes of cream were served, and it was found that it was still $\frac{1}{8}$ full. How many dishes remained?

45. A merchant purchased 460 rugs for $55,200. He sold one-fourth of them at $150 each, 15 at $165 each, and the remainder at $180 each. Find the total selling price and the total gross profit.

46. A watch is 30 sec slow at noon on Dec. 7 and 12 sec slow at noon on Dec. 13. How much does the watch gain or lose per day?

47. Over a given air base the temperature decreased 1.98°C for each 1,000 ft of increase in elevation. If the ground temperature was 20°C, find the temperature at the following heights above the ground: (*a*) 4,400 ft; (*b*) 12,000 ft; (*c*) 26,000 ft.

48. Change each of the following common fractions to a decimal fraction of four decimal places: (*a*) $\frac{22}{7}$, (*b*) $\frac{3}{13}$, (*c*) $\frac{27}{4}$, (*d*) $\frac{5}{11}$, (*e*) $\frac{9}{50}$.

49. When one man works 1 hr, we say that 1 man-hr of work has been performed. If it takes 25 men 5 days working 8 hr per day to make a certain assembly jig, how many man-hours does the construction of the jig require?

50. If 465 men "lay off" for 4 hr, how many man-hours of labor have been lost?

51. A crew laid and finished 800 sq ft of sidewalk in a day. The crew consisted of four laborers at $12 per day and one mixer operator at $18 per day and one finisher at $18.50 per day. What was the cost of labor per square foot?

52. The weekly output of pieces completed by three welders was as follows:

	Welder A	*Welder B*	*Welder C*
Mon.	216	232	137
Tues.	147	269	192
Wed.	219	241	87
Thur.	145	176	106
Fri.	218	108	56
Sat.	183	228	113

At 7¢ per piece, find the weekly pay roll for each welder.

53. A bomber left an air base at 0730 for a target 850 miles distant. The ground speed out was 290 mph, and the speed in was 325 mph. If 22 min was spent over the target, at what time did the bomber return to its base?

54. On a test run a drill engine uses $1\frac{13}{20}$ gal of gasoline in an hour. How long would 12 gal last?

55. A housewife made the following purchases at the corner grocery:

2 lb bacon at 90¢ a pound,
5 lb sugar at 10¢ a pound,
4 cans tomatoes at 17¢ a can,
3 boxes frozen Lima beans at 39¢ a box,
1 loaf bread at 21¢ a loaf
$4\frac{1}{2}$ doz oranges at 36¢ a dozen.

What amount of change did she receive from a ten-dollar bill?

56. If a man's income is $4,400 per year and his expenses are $3,160, what part of his income does he save?

57. An advertisement for acetate lamp shades reads as follows:

Model	*Price*	*Model*	*Price*
8-in. boudoir...........	$1.00	14-in. table.............	$2.15
9-in. clip...............	1.35	15-in. table.............	2.39
12-in. clip...............	1.89	19-in. floor.............	2.95

What would be the total cost of an order for a half dozen of each model?

58. If 24 doz pairs of nylons cost $528.56, how much do they cost per pair?

59. A gallon contains 231 cu in. Alcohol is 0.84 as heavy as water. What volume of alcohol will weigh as much as 2 gal of water?

60. A Red Cross canteen operating on a 24-hr basis served 7,350 transient patients and 5,246 servicemen on transcontinental trips over a period of 2 years. What was the average number of patients and servicemen served per week?

61. A suit of clothes requires $4\frac{3}{4}$ yd of cloth. What is the value of suits made from 24 yd if each suit is worth $45.95?

62. Add the following truck loadings, and check the answer by casting out nines: 2,358, 5,643, 4,734, 3,786, 2,849, 5,237, 4,385, 2,647, 2,369, 5,308, 2,704, 3,748.

Self-Test*

Time: 50 Minutes

(Score 10 points for each problem correctly solved)

1. Receipts for seven home basketball games at a certain college were as follows:

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* At the end of each chapter throughout this book there is given a "self-test" that covers much of the material taken up in that chapter. After completing

Dec. 3	$2,714.40
Dec. 12	1,922.75
Dec. 18	1,753.90
Dec. 23	1,068.25
Jan. 6	2,997.15
Jan. 13	2,542.35
Jan. 22	4,100.60

(*a*) Find the total receipts for the seven games. (*b*) How much greater were the receipts for the final game than for the game just prior to the Christmas vacation?

2. At $4.80 per half dozen, how many more golf balls can be purchased for $9.75 than can be bought at the rate of $0.95 each?

3. How many significant figures are there in each of the following measurements obtained in a physics laboratory: (*a*) 300.03, (*b*) 0.00001, (*c*) 939.21, (*d*) 4,636,000, (*e*) 2.2002?

4. A banker used the method of casting out nines to check the following deposits·

$2,369.11,
4,824.03,
382.44,
7,731.90,
45.67,
394.06,
1,078.32.

(*a*) Find the total receipts. (*b*) Check by casting out nines, and state the "excess" arrived at in the check.

5. Find the smallest number exactly divisible by each of the numbers 63, 24, 98, and 210. What is this number called?

6. A merchant has four lots of rope in lengths of 54 ft, 90 ft, 126 ft, and 162 ft. He wishes to sell the entire supply as automobile tow ropes. Into what lengths should the ropes be cut to obtain the longest uniform tow ropes without waste?

7. Find the value of 92.479 divided by 0.92, rounding off the quotient to two decimal places.

8. A man who owned $\frac{1}{5}$ of the shares of a corporation sold $\frac{1}{3}$ of his holdings to his brother. What fraction of the shares did his brother receive?

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the work of a chapter, the student should take the self-test and score his own paper. If he scores fewer than 80 points, he has probably failed to master sufficient material for success in the course. It is suggested that when a problem is missed, pertinent text material be reviewed immediately.

9. The number of board feet in an oak beam was calculated from the product $22\frac{2}{5} \times 1\frac{7}{8} \times 24$. What was the cost of the beam at $0.045 per board foot?

10. Find each division pay roll and the total pay roll for the following 532 employees:

Division	*Number of Employees*	*Hours Worked per Employee*	*Rate per Hour*
Management.	76	43	$1.80
Advertising.	145	45	1.72
Sales.	262	48	1.20
Collections.	49	40	1.15

Chapter II

SIMPLE ALGEBRAIC OPERATIONS

Introduction. Skill and proficiency in the use of the simple arithmetic processes reviewed in the last chapter are necessary in almost every walk of life—most certainly in the realms of business and industry. These arithmetic processes consist mainly, as we have seen, of the addition, subtraction, multiplication, and division of numbers involving the digits 0, 1, 2, . . . , 9 and are sufficient to enable us to work many kinds of problems. But arithmetic alone is not adequate to solve many of the important problems which arise in the financial, industrial, and business fields. Time must be saved, and information gained from specific sources must be generalized. For these reasons and others that will become apparent later, a study of the fundamentals of *algebra* is important as a medium through which business mathematics and its applications may be approached.

20. The Meaning of Algebra. For practical purposes algebra may be thought of as a continuation of arithmetic in which symbols (usually letters) are used to represent quantities. For example, in problems dealing with money at interest, the principal is usually represented by the letter P, the interest by I, the rate of interest by r or i, and the time involved by the letter t. Sometimes, too, the representation of a certain quantity takes the form of some combination of letters and symbols. Thus, v^n often stands for the present value of an amount at compound interest for n years; $s_{\overline{n|}i}$* usually designates the amount of an annuity of $1 per year for n years with an interest rate of i. In short, the letters and symbols of algebra furnish us with a kind of shorthand, making possible a great conservation of time and labor.

When letters (or other symbols), or letters and numbers, are combined by symbols of operation, they form an *algebraic expres-*

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* The symbol $s_{\overline{n|}i}$ is single and indivisible, and the n and i cannot be read independently. The symbol is read "s angle n at i."

sion; and the parts that are separated by plus or minus signs are called *terms* of the expression. An algebraic expression containing only one term is called a *monomial;* if it contains two terms, it is called a *binomial;* and if it contains more than two terms, it is ordinarily (but not always) called a *polynomial.*

A statement that one expression is equal to another is called an *equation.* The two *members* of the equation are separated by the symbol $=$. Thus, $2x + 5 = 11$ is an equation expressing the equality of the quantities $2x + 5$ and 11, the two members of the equation. This equation is an example of a *conditional* equation, since it is true only when x has the value 3. An equation such as $3x + 5x = 8x$, is called an *identity,* since it is true for all values of x for which the equation is defined. Finding the *solutions* of equations, that is, values for which the equation is true, plays an important role in many business and industrial problems.

As in the case of arithmetic, the fundamental operations with algebraic quantities are addition, subtraction, multiplication, and division, and these operations are indicated by the usual signs. Thus, $a + b$ means that b is added to a; $a - b$ means that b is subtracted from a. Although multiplication may be indicated by the signs $\times$ or $\cdot$, often no special sign is needed when the meaning is clear. Thus, the product 3 times a is usually written simply as $3a$. The sign $\div$ indicates division, but often the division is indicated in fractional form. Thus the quotient $a \div b$ may be written $\frac{a}{b}$, or a/b. The product of the two quantities a/b and c/d may be written $\frac{a}{b} \cdot \frac{c}{d}$ or $\frac{ac}{bd}$.

Each of the numbers appearing in an indicated product of two or more numbers is called a *factor.* For example, in the binomial $2S + 4RT$, the first term is the product of the two factors 2 and S, and the second term is the product of the three factors 4, R, and T. In a term like $2S$, the 2 is said to be the numerical *coefficient* of S. In later sections we shall discuss the process of *factoring* an algebraic expression by writing it as the product of simple factors.

Although it is possible to express in words the various rules of computations and facts concerning business problems, they may be more precisely and compactly stated in mathematical terms. For example, in words we say that "profit equals the selling price less the cost." In symbols, this statement may be expressed as follows: Let P = Profit, C = Cost, and S = Selling price. Then, the formula for profit in terms of cost and selling price may be written $P = S - C$. Since there are three letters in this formula (equation), any one of them can be found when the other two are known. Thus, if an article costing \$60 is sold for \$72.25, we merely have to substitute in the formula and write $P = \$72.25 - \$60 = \$12.25$. A more detailed treatment of the use of formulas and the setting up and use of equations will be given in later sections.

As we have pointed out, symbolic expressions are the bases upon which algebra rests. For this reason we shall begin with the problem of translating written statements and expressions into mathematical terms. This ability to express in symbols specific conditions stated in words is of the greatest importance, for it is the first step in the logical solution of a problem after the problem is clearly understood. For example, in a certain problem the following statement appeared: "The total weight of three different shipments was 680 pounds." How may we express this statement in symbols? Since there are three different weights, we may let some letter (different for each) stand for each of them. Thus, we may let x stand for the first weight, y the second, and z the third. The resulting mathematical statement would then be $x + y + z = 680$. Of course, the choice of symbols or letters is entirely optional, and the sum of the three weights could just as well be written $a + b + c$, or $W + M + S$, or $W_1 + W_2 + W_3$, or in terms of any other symbols that were convenient.

Illustrations: Write the following in algebraic symbols:

1. If one orange costs N cents, what do 18 oranges cost? Answer: $18N$ cents. We observe that $18N$ means 18 times N. Sometimes this product is written $18 \times N$, where the $\times$ indicates multiplication. Other ways of indicating multiplication are to use a dot ($\cdot$) between the factors or to enclose each factor within parentheses. Thus, 18 times N, $18N$, $18 \cdot N$, and $(18)(N)$ all mean the same thing.

2. Mr Jacobs sold a car for \$750 and lost x dollars. How much did the car cost? Let C = cost. Then, since the car cost x dollars more than it was sold for, $C = 750 + x$.

3. If N cans of dog food cost C cents, how much do 5 cans cost? Since N cans cost C cents, 1 can would cost $C \div N$, or C/N cents. Hence, 5 cans would cost 5 times C/N cents, or $5C/N$ cents. (Until the student becomes skillful in the use of symbols, he may find it helpful to compare the symbolic expression with the same statement when numbers have been substituted. Thus, in translating the given statement, one might think, "If 2 cans cost 8¢ . . . , to find the cost of 1 can we must divide . . . ," and so forth.)

4. Twice a number plus a second number equals 13. Let $a =$ one number, and $b =$ second number. Then the problem states that $2a + b = 13$.

PROBLEMS

1. Let a, b, and c be three given numbers. Write in symbols: (*a*) The sum of the three numbers. (*b*) The sum of the first two minus the last. (*c*) The product of the three numbers. (*d*) The quotient of the first divided by the second. (*e*) The quotient of the first two, minus the third. (*f*) The product of the last two, minus the first. (*g*) The product of the three numbers, plus the product of the last two. (*h*) The sum of the first two numbers, divided by the third.

2. Express in algebraic symbols: (*a*) The product of P times r times t. (*b*) The quotient of A times R, divided by n. (*c*) Half of the sum of $x + y$. (*d*) The quotient of the sum of c and d divided by the difference obtained when d is subtracted from c.

3. If 12 apples cost x cents, what does 1 apple cost?

4. If a boy can walk 5 miles in t hr, how far can he walk in 1 hr?

5. If an automobile makes 20 miles on the gallon, how far can it travel on y gal?

6. If a Girl Scout can bicycle x mph, how many hours does it take her to go 1 mile?

7. An article is sold for \$90. The profit realized is $\frac{2}{5}$ of the cost. Let x stand for the cost, and write an equation expressing the relationship Cost + Profit = Selling price.

8. Write an equation (formula) which states that simple interest I equals principal P times rate r times time t.

Write the following in algebraic symbolism:

9. Tom is a years old. Ruth is b years old, or twice as old as Tom.

10. John is z years old. His age in 5 years will be 15.

11. T. C. Smith sells brooms for w cents each. He reduces the price 15¢ and sells them for z cents.

12. Brown's Haberdashery will sell 1 belt for r cents but will sell 3 for q cents, d cents off on each belt.

13. A salesman gets a commission of $\frac{1}{12}$ on his sales s in dollars. His commission for the month is C dollars.

14. The Crystal Products, Inc., have net earnings of E dollars. Their bonus payments total B dollars. They pay a tax of $\frac{1}{3}$ on the net less the bonus. The tax is T dollars and C cents.

Solve the following problems in algebraic language:

15. Dick has y dollars, and Harry has 4 times as many. How many does Harry have? How many dollars do both have together?

16. How much money does Jones have left out of x dollars if he spends y cents?

17. An automobile travels y miles in t hr. How far does it travel in 1 hr? In T hr? In 4 hr? In k hr 30 min?

21. Positive and Negative Numbers. In arithmetic we dealt with positive numbers only. Positive numbers include the *integers* 1, 2, 3, 4; the *positive rational fractions*, such as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{23}{50}$; and the *positive irrational numbers*, such as $\sqrt{2}$, $\sqrt{5}$.* All these numbers may be thought of as having a value greater than zero. When they are added, multiplied, and divided, the results are always positive. In subtraction, the result is positive if the smaller number is subtracted from the larger one; but arithmetic does not enable us to subtract a larger from a smaller number.

The introduction of negative numbers in the development of algebra made possible the subtraction of larger from smaller numbers. Negative numbers may be thought of as being less than zero and are written with a minus sign prefixed. Thus, negative 6 is written -6; negative $2x$ is written $-2x$. To subtract 4 from 2, we write 2 minus 4, or $2 - 4$, which equals -2.

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* A *rational number* is defined as one that may be expressed as the quotient of two integers; an *irrational number* is one that cannot be expressed as the quotient of two integers.

The numerical value of a number, or its value taken without regard to sign, is called its *absolute value.* Thus, the absolute values of -6, $+8$, -75, $+2.45$, and -0.06 are 6, 8, 75, 2.45, and 0.06, respectively. A debt of \$400 might be regarded as $-\$400$, and a profit or perhaps an income of the same amount might be recorded as $+\$400$. The absolute values of the two items would be exactly the same, although their signs are different.

22. The Real Number System. The numbers of arithmetic, the negative numbers, and the number zero, with their rules of operation and relation, form the system of numbers known as the *real number system.* These numbers have certain properties which are for the most part self-evident and which are the basis upon which algebra rests. These fundamental assumptions are as follows:

1. The sum of two numbers is the same in whatever order the numbers are added. Thus,

$$6 + 4 = 4 + 6,$$

$$a + b = b + a \qquad \text{(Commutative Law for Addition).}$$

2. The sum of three or more numbers is the same no matter how they are grouped. Thus,

$$2 + 5 + 6 = (2 + 5) + 6 = 2 + (5 + 6),$$

$$a + b + c = (a + b) + c = a + (b + c)$$

(Associative Law for Addition).

3. The product of two numbers is the same in whatever order the numbers are multiplied. Thus,

$$3 \cdot 4 = 4 \cdot 3,$$

$$ab = ba \qquad \text{(Commutative Law for Multiplication).}$$

4. The product of three or more numbers is the same no matter how they are grouped. Thus,

$$3 \cdot 4 \cdot 5 = (3 \cdot 4) \cdot 5 = 3 \cdot (4 \cdot 5),$$

$$abc = (ab)c = a(bc) \qquad \text{(Associative Law for Multiplication).}$$

5. The product of a number by the sum of two or more numbers is equal to the sum of the products of the first number multiplied by each of the numbers of the sum. Thus,

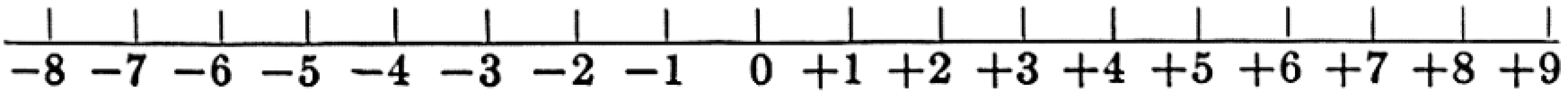
$$2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4,$$

$$a(b + c) = ab + ac \qquad \text{(Distributive Law).}$$

6. If the same operation is performed on equal quantities, the results are equal. Thus, if $x = y$, then $x + a = y + a$; and $x - a = y - a$. If $x = y$, then $ax = ay$; and $x/a = y/a$, provided that a does not equal zero.

7. If $a = b$ and $b = c$, then $a = c$.

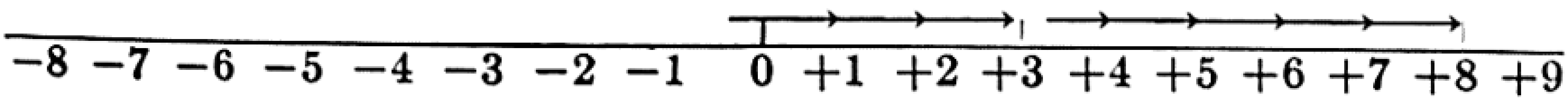
It is often convenient to represent the real numbers graphically by associating them with points along a chosen straight line. Positive numbers are directed along the line to the right of an arbitrarily chosen point 0, whereas negative numbers are laid off to the left of the zero point.



23. Addition. If we agree that "plus is to the right" and "minus is to the left," addition and subtraction of numbers, whether they be positive or negative, can be illustrated graphically, using a numbered scale.

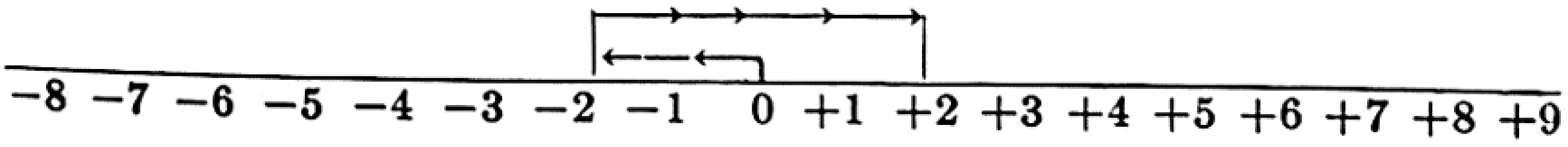
Examples:

1. $3 + 5 = ?$



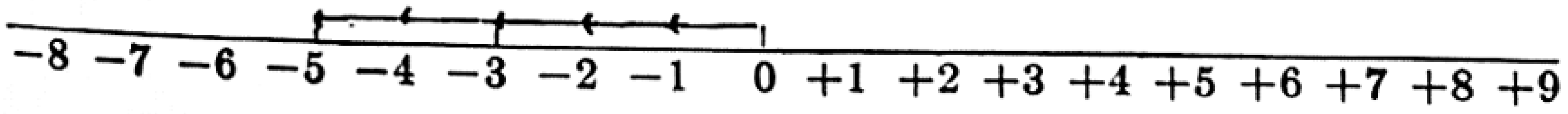
Starting at zero, we reach +3 by moving 3 units to the right. The plus sign in front of the 5 means that we must move 5 more units to the right, reaching +8, which is the correct sum.

2. $-2 + 4 = ?$



Starting at zero, we reach −2 by moving 2 units to the left. From this point we move 4 units to the right, arriving at +2, the sum of −2 and +4.

3. $-3 - 2 = ?$



Starting at zero, we obtain the correct sum, -5, by moving 3 points to the left for the -3, and then 2 more points to the left for the -2.

The above examples illustrate the addition of numbers of like signs (Examples 1 and 3) and of unlike signs (Example 2). The ideas involved in the process of adding lead quite naturally to the following rules of addition:

1. *To add two numbers having like signs, add their absolute values, and prefix the common sign.*

2. *To add two numbers having unlike signs, take the difference of their absolute values, and prefix to it the sign of the number having the larger absolute value.*

When three or more numbers are to be added, the above rules may be applied by adding first the positive numbers and then the negative numbers, obtaining two numbers whose sum will be the total algebraic sum required.

PROBLEMS

Add: (1) $\begin{array}{r}-3\\ \underline{-5}\end{array}$ (2) $\begin{array}{r}-10\\ \underline{4}\end{array}$ (3) $\begin{array}{r}4\\ \underline{-7}\end{array}$ (4) $\begin{array}{r}-8.9\\ \underline{-0.2}\end{array}$ (5) $\begin{array}{r}-83\\ \underline{-25}\end{array}$

(6) $\begin{array}{r}673\\ \underline{-921}\end{array}$ (7) $\begin{array}{r}-445\\ \underline{581}\end{array}$ (8) $\begin{array}{r}-341.8\\ \underline{15.3}\end{array}$ (9) $\begin{array}{r}0.007\\ \underline{-0.121}\end{array}$ (10) $\begin{array}{r}0.6255\\ \underline{-4.4107}\end{array}$

(11) $\begin{array}{r}3\frac{1}{2}\\ \underline{-2}\end{array}$ (12) $\begin{array}{r}-\frac{3}{8}\\ \underline{\frac{1}{5}}\end{array}$ (13) $\begin{array}{r}-17\frac{7}{8}\\ \underline{5\frac{1}{3}}\end{array}$ (14) $\begin{array}{r}-\frac{2}{5}\\ \underline{-\frac{9}{4}}\end{array}$ (15) $\begin{array}{r}-3\frac{2}{5}\\ \underline{-7\frac{1}{8}}\end{array}$

(16) $\begin{array}{r}-9\\ \underline{1\frac{5}{6}}\end{array}$ (17) $\begin{array}{r}\frac{4}{7}\\ \underline{99\frac{1}{5}}\end{array}$ (18) $\begin{array}{r}-0.0007\\ \underline{0.0006}\end{array}$ (19) $\begin{array}{r}5{,}892{,}617\\ \underline{-5{,}892{,}716}\end{array}$

(20) $\begin{array}{r}327\\ \underline{-\frac{2}{3}}\end{array}$ (21) $\begin{array}{r}-\frac{11}{12}\\ \underline{\frac{11}{12}}\end{array}$ (22) $\begin{array}{r}\frac{4}{11}\\ \underline{0}\end{array}$

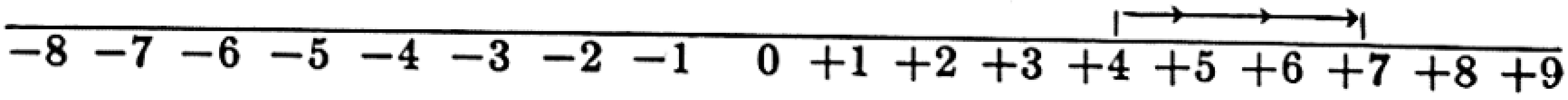
(23) $\begin{array}{r}5\\ -3\\ 7\\ \underline{-10}\end{array}$ (24) $\begin{array}{r}6\\ -15\\ 11\\ \underline{-17}\end{array}$ (25) $\begin{array}{r}-1.17\\ -2.34\\ 0.63\\ \underline{-9.10}\end{array}$ (26) $\begin{array}{r}6\frac{7}{8}\\ -2\frac{1}{4}\\ -3\frac{3}{4}\\ \underline{1\frac{5}{8}}\end{array}$

(27) $-3 + 2.1$,
(28) $6 - 8.2$,
(29) $-6\frac{2}{3} - 2\frac{3}{4}$,
(30) $19.6 - 8\frac{1}{2}$,
(31) $396 - 397$,
(32) $8 - 3 + 2 + 6 - 7 - 9$,
(33) $-2 - 4 - 9 + 6 - 3 + 19$,
(34) $6.25 + 3.46 - 10.06 + 2.96$,
(35) $-1.2 - 2\frac{1}{2} - 8.2 + 11.9$,
(36) $-\frac{3}{4} + 1\frac{1}{2} + 2\frac{3}{8} - 3\frac{1}{8}$,
(37) \$92.67 + \$18.34 − \$2.20 − \$67.92 − \$13.80,
(38) $21\frac{1}{3} - 62\frac{5}{7} + 2\frac{5}{6} - 18\frac{1}{2}$,
(39) $-48.72 + 16.02 + 16.60 + 16.09$,
(40) $0 - 9 - 3\frac{3}{4} + 12$.

24. Subtraction. We may think of subtraction as the inverse of addition. Thus when we subtract 4 from 7 and obtain 3, we have found a number which when added to 4 yields 7. Graphically, this amounts to starting at 4 and counting off the number of units to reach 7.

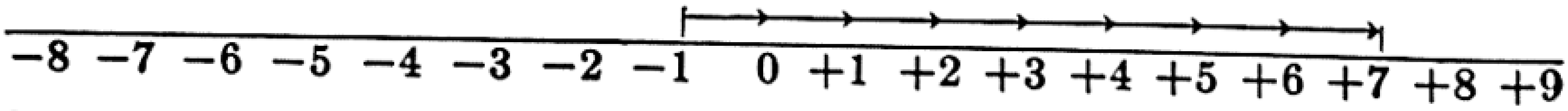
Examples:

1. $(+7) - (+4) = ?$



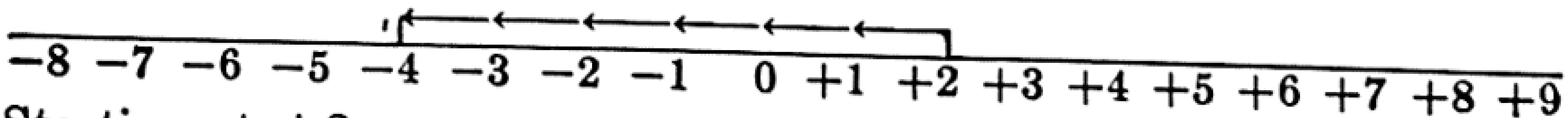
Starting at $+4$, we must count 3 units *to the right* to reach $+7$. Therefore, $7 - 4 = 3$.

2. $(+7) - (-1) = ?$



Starting at -1, we must count off 8 units *to the right* to reach $+7$. Therefore, $(+7) - (-1) = +8$.

3. $(-4) - (+2) = ?$



Starting at $+2$, we must count off 6 units to the left to reach -4. Therefore, $(-4) - (+2) = -6$.

These examples illustrate subtracting positive numbers (Examples 1 and 3) and negative numbers (Example 2). A study of the principles involved will show that the following rule for subtraction holds:

To subtract one number from another, change the sign of the number to be subtracted (the subtrahend), and proceed as in addition.

PROBLEMS

Subtract: (1) $\begin{array}{r}5\\ \underline{7}\end{array}$ (2) $\begin{array}{r}10\\ \underline{23}\end{array}$ (3) $\begin{array}{r}-76\\ \underline{-17}\end{array}$ (4) $\begin{array}{r}-21\\ \underline{14}\end{array}$ (5) $\begin{array}{r}113\\ \underline{-107}\end{array}$

(6) $\begin{array}{r}79\\ \underline{18}\end{array}$ (7) $\begin{array}{r}37.06\\ \underline{-14.89}\end{array}$ (8) $\begin{array}{r}161.7\\ \underline{293.9}\end{array}$ (9) $\begin{array}{r}-0.0172\\ \underline{0.6122}\end{array}$ (10) $\begin{array}{r}-14\\ \underline{0}\end{array}$

(11) $\begin{array}{r}-\frac{7}{8}\\ \underline{-\frac{5}{8}}\end{array}$ (12) $\begin{array}{r}\frac{7}{5}\\ \underline{\frac{9}{5}}\end{array}$ (13) $\begin{array}{r}-\frac{4}{7}\\ \underline{\frac{2}{5}}\end{array}$ (14) $\begin{array}{r}13\frac{2}{3}\\ \underline{19\frac{1}{6}}\end{array}$ (15) $\begin{array}{r}0\\ \underline{-\frac{9}{11}}\end{array}$ (16) $\begin{array}{r}19\frac{4}{9}\\ \underline{-22\frac{5}{18}}\end{array}$

(17) $\begin{array}{r}-9\frac{21}{23}\\ \underline{7\frac{2}{3}}\end{array}$ (18) $\begin{array}{r}\frac{9}{15}\\ \underline{1.07}\end{array}$ (19) $\begin{array}{r}-92.3\\ \underline{4\frac{2}{7}}\end{array}$ (20) $\begin{array}{r}342\frac{3}{7}\\ \underline{-38.25}\end{array}$ (21) $\begin{array}{r}6{,}394.29\\ \underline{-415.89}\end{array}$

(22) $\begin{array}{r}-2{,}234.03\\ \underline{-1{,}164.09}\end{array}$ (23) $\begin{array}{r}-1{,}111.11\\ \underline{1{,}111.11}\end{array}$ (24) $\begin{array}{r}3{,}492.896\\ \underline{16.007}\end{array}$ (25) $\begin{array}{r}192\frac{5}{16}\\ \underline{-113\frac{1}{6}}\end{array}$

(26) $\begin{array}{r}-436\frac{2}{5}\\ \underline{-435\frac{1}{7}}\end{array}$ (27) $\begin{array}{r}400\frac{2}{3}\\ \underline{462\frac{1}{8}}\end{array}$

(28) $-735\frac{1}{6} - (+102\frac{3}{5})$,
(29) $7 - (-3)$,
(30) $19 - (+2)$,
(31) $-5 - (-9)$,
(32) $-72 - (+49)$,
(33) $82\frac{1}{3} - (+92\frac{2}{3})$,
(34) $16.3 - (-16.3)$,
(35) $-26.7 - (-27.8)$.

25. Addition and Subtraction of Like Terms. In algebra, like quantities may be added or subtracted as in arithmetic. Thus, $3x$, $4x$, and $7x$ are *like terms*, and the sum of these three terms is $14x$. $\frac{9}{7}ab$ and $\frac{4}{7}ab$ are like terms, and we may subtract the $\frac{4}{7}ab$ from the $\frac{9}{7}ab$ for a difference of $\frac{5}{7}ab$. So long as the terms are expressed in the same unit, we may proceed in algebra as we did in arithmetic. This indicates that if an expression contains terms having different signs of operation, the *proper order of operations* is to perform multiplication and division first, then addition and subtraction. Thus, $6a \div 2 + 8a \times 2 - 3a \times 4 = 3a + 16a - 12a = 7a$.

Illustrations:

1. Add:

$$\begin{array}{rrrr} 4x & 9xy & -12abc & 0.06P \\ -5x & -2xy & -15abc & 0.03P \\ 8x & 13xy & -\ \ 4abc & 1.04P \\ \hline 7x & 20xy & -31abc & 1.13P \end{array}$$

2. Subtract:

$$\begin{array}{rrrr} 23rt & 1.5C & -32xy & -24w \\ 3rt & -0.5C & 11xy & -\ \ 6w \\ \hline 20rt & 2.0C & -43xy & -18w \end{array}$$

If the terms to be added are *unlike*, we may indicate the addition in the following manner:

$$\begin{array}{l} 33x \\ \ \ 2y \\ \hline 33x + 2y \end{array} \qquad \begin{array}{l} 2ab \\ -7E \\ 6P \\ \hline 2ab - 7E + 6P \end{array} \qquad \begin{array}{l} 200Rt \\ \ \ 35L \\ -16ab \\ \hline 200Rt + 35L - 16ab \end{array}$$

PROBLEMS

Add the following by combining like terms:

1. $3x$, $4x$, $12x$,
2. $5y$, $10y$, $3y$,
3. $2x$, $-5x$, $4x$,
4. $6w$, $-9w$, w,
5. $7w$, $-10w$, $-17w$,
6. $-6z$, $-z$, $-4z$,
7. $4xz$, $2xz$, $7xz$,
8. $17xz$, $-9xz$, xz,
9. $3cd$, $-3cd$, $-2cd$,
10. x^2y, $3x^2y$, $25x^2y$,
11. M_x, $5M_x$, $12M_x$,
12. $0.40x$, $-0.50x$, $0.06x$,
13. $-6.25x$, $-0.45x$, x,
14. $\frac{1}{4}a$, $\frac{1}{2}a$, $\frac{3}{4}a$.

Simplify the following by uniting like terms:

15. $a - 3a + 2a$,
16. $2b - 3a + 7b - 5a$,
17. $8 + 2x - 3y + 7 + y$,
18. $6xy + 7x - 10y - xy - x - y$,
19. $\frac{1}{5}y^3 - \frac{1}{2}x^2 + 35y^3 + 4x^2$,
20. $1.5z + 2.4w - 19.6z + 4w$,

21. $vd_x,\ 6vd_x,\ -3vd_x,$
22. $2a^2b - 5ab^2 - 4ba^2 + 3b^2a,$
23. $7.9r + 0.6i - r - 12i,$
24. $99D - 6t - 0.05D,$
25. $\frac{9}{11}P - \frac{3}{22}S + \frac{9}{33}S - \frac{6}{11}P,$
26. $L - a + 16n - a - 9n + \frac{1}{2}a,$
27. $35P - \frac{5}{6}r - \frac{9}{8}P + 7 + \frac{4}{3}r - \frac{3}{5},$
28. $M_1 - 2N_1 + 3M_1 - N_1.$

26. Symbols of Grouping. Several parts of an algebraic expression are often grouped together to indicate that they are to be considered as a single number. Thus, in the expression $50 - (10 + 5)$, the parentheses indicate that the 10 and the 5 are to be considered as the single number 15 when subtraction is made from 50. Likewise, $A - (P + I)$ means that the P plus the I is to be treated as a single quantity which is to be subtracted from A.

The symbols used for grouping are: the parentheses ( ); the brackets [ ]; the braces { }; and the vinculum $\overline{\qquad}$. These are the signs of aggregation that are in common use and may be employed wherever grouping is desired, subject only to the laws of signs that result from the fundamental assumptions of algebra. We have different symbols of grouping to enable us to group terms which are already grouped within another symbol. Thus, we write $a - [a + b - (a + c)]$, to indicate that the $a + c$ within the parentheses is treated as a single quantity within the larger group enclosed by brackets. The following examples will serve to show what happens when parentheses or other symbols of grouping are removed and what must be done when such symbols are inserted.

Examples:

1. $a + (b + c) = ?$

The two numbers within parentheses are to be added together, treated as one number, and then added to a. Since it is a problem in addition, we may set the numbers down as follows and add:

$$\begin{array}{r} a \qquad\quad \\ b + c \\ \hline a + b + c \end{array}$$

We observe that the parentheses have been removed and the signs of the numbers within parentheses remain unchanged. Furthermore, since $a + (b + c) = a + b + c$, we see that parentheses may be inserted without affecting the signs of the terms.

2. $a + (b - c) = ?$

Proceeding as in Example 1, we find

$$\begin{array}{l} a \\ \phantom{a+{}} b - c \\ \hline a + b - c \end{array}$$

Again we see that parentheses preceded by a plus sign may be removed without affecting the signs of the numbers that were within the parentheses, and that terms may be enclosed within symbols of grouping preceded by a plus sign without changing their signs.

3. $a - (b + c) = ?$

Since this is a problem in subtraction in which the quantity within parentheses is to be subtracted from a, we may set the numbers down as follows and subtract in the usual manner, being careful to change the signs of the subtrahend:

$$\begin{array}{l} a \\ \phantom{a-{}} b + c \\ \hline a - b - c \end{array}$$

We see that when the parentheses have been removed, the signs of the numbers within the parentheses are changed. The next example will confirm this change in signs for negative numbers within the parentheses.

4. $a - (-b - c) = ?$

Subtracting as in the previous example, we find

$$\begin{array}{l} a \\ - b - c \\ \hline a + b + c \end{array}$$

Hence, $a - (-b - c) = a + b + c$.

The above examples are verifications of the rules of grouping that must arise, of course, out of the ideas involved in the fundamental operations of algebra. We may state the rules as follows:

1. *Parentheses or other symbols of grouping that are preceded by a* plus *sign may be removed by rewriting the inclosed terms with their original signs.*

2. *Parentheses or other symbols of grouping that are preceded by a* minus *sign may be removed by changing the sign of each term within the original group.*

3. *Parentheses or other symbols of grouping may be inserted in accordance with the rules for removing symbols (see Illustration 2).*

Illustrations:

1. Remove the symbols of grouping, and combine like terms:

(*a*) $3 - [4 + (5 - 3) - (7 - 4)]$

$= 3 - [4 + 5 - 3 - 7 + 4]$ (Notice that the signs of the 5 and the 3 within the first parentheses remain unchanged whereas the signs of the 7 and 4 within the second parentheses have been changed. Why?)

$= 3 - 4 - 5 + 3 + 7 - 4$ (Notice that the signs of all numbers that were within the brackets have been changed. Why?)

$= 0.$

(*b*) $6a - [5a - (b - 2a) + 4b]$

$= 6a - [5a - b + 2a + 4b]$ (Removing the parentheses)

$= 6a - 5a + b - 2a - 4b$ (Removing the brackets)

$= -a - 3b.$

2. Place the last three terms of the following expressions in parentheses preceded by a minus sign:

(*a*) $x - 2y + 3z - 4w$

$= x - (2y - 3z + 4w)$ (Notice that each term placed within the parentheses has its sign changed. Why?)

(b) $24 - 4P + 2R - 7PR + I$
$= 24 - 4P - (-2R + 7PR - I)$ (What effect has the minus sign before the parentheses upon the terms placed within?)

PROBLEMS

Simplify the following expressions by removing signs of grouping and combining like terms:

1. $5r + (r - 4)$,
2. $6x + (x - 3)$,
3. $14 + (2i + 6)$,
4. $-2 + (-x + 2)$,
5. $3P - (6r + 5)$,
6. $4y - (3x - 3)$,
7. $9P - 6 - (3P + 2)$,
8. $5x - 2y - (2x - 3y)$,
9. $6 + a - (3a + 2)$,
10. $6 - A - (-A + 4)$,
11. $6t + (4.2t - 1) - 1$,
12. $5i + (6.7i + 1) - 2$,
13. $-(9 + 6x - 2y)$,
14. $-(2 - 3i + 2k) + 3$,
15. $(3x + 2y - 3z) - (y + 4x - 3z)$,
16. $(5k - 6e + 7k) + (4e - 6k - 2k)$,
17. $-(ab - b^2 - a^2) + (3a^2 - 2b^2 - 4ab)$,
18. $1 - [1 - (1 - 1)]$,
19. $8p - (3m + 2n + 9m - 3n)$,
20. $3a - [a + b + (b - a) - 3]$,
21. $(6 + 2y) + (9 - 2y)$,
22. $5 + [6a - (b - 2a - \overline{4a + 2})]$,
23. $(13r + 2n) - (4r - 7n)$,
24. $pq - \{p - [q + (3 - pq)] + p\}$,
25. $5a + [2L + 3a - (4a + L)]$,
26. $2y + (6 - \overline{y + 3} + y)$,
27. $2x + [4x - (x - 7x) - 4x] + 3$,
28. $1 - \{1 - [1 - (1 - \overline{1 - 1})]\}$,
29. $5 - \{2a - [5a - (7a + 3b - 4) - 8a]\}$,
30. $3 + [-2 + 4 - (2 + 1)] - (3 - 4)$,

31. $5r - \{(3r - i) - [2i - (3 + 2r) - 4i] + \overline{r - i}\}$,
32. $5 - \{5 + [4 - (3 - 1)] - 5\}$.

Place the last three terms of the following expressions in parentheses preceded by a minus sign. Remove the parentheses to check your work.

33. $2x^2 - 4x + 3y - 2$,
34. $a + b + 1 - m - n$,
35. $k + P - Prt + D$,
36. $C - D_x - D_y - D_z$,
37. $10 + 1 + 1 + 1$,
38. $9z - w - x - 3$,
39. $1 + 3P - 4Q + R$,
40. $rs + xy - 3 - 4z$,
41. $y^2 - y + 6y^2 - 4x + 7$,
42. $-x - y - z - w + 10$,
43. $(P + S) - 4r + 3n - 6q$,
44. $10 - 4q + 3p - 7r$,
45. $(2x + y) + 5 - z + 3w$,
46. $a - 2e - 3n + 6$,
47. $-a - b - c - d$,
48. $9m_1 + 3m_2 - m_3 - 6m_4$,
49. $7 - x + y$,
50. $-3x + 2y - 7$.

27. Positive Integral Exponents. One of the most useful symbols of algebra is the exponent. If, for example, we wished to multiply 4 by itself two times, we could write the product as $4 \cdot 4 \cdot 4$. But we may write the indicated multiplication more simply as 4^3, where the 3 is placed to the right and a little above the 4 to indicate how many times the latter is to be taken as a factor. The product $1.06 \cdot 1.06 \cdot 1.06 \cdot 1.06 \cdot 1.06$ is written simply $(1.06)^5$, and is read "1.06 to the fifth power." In general, if n is a positive whole number, a^n means $a \cdot a \cdots a$ (n factors).

A knowledge of exponents would be valuable if only for the purpose of writing certain products in a concise and simplified form. But as we shall see in a later chapter, exponents, both integral and fractional, positive and negative, obey certain laws of operation which make them indispendable to a working knowledge of the mathematics that enters into almost all business problems.

We shall consider here only a few simple problems involving *positive integral exponents* as background for the work in multiplication and division that follows. Study of the laws and the meaning of fractional, zero, and negative exponents will be reserved for a later chapter.

Illustration 1: $a^3 \cdot a^4 = ?$

Writing the indicated products in extended form, we have

$$\begin{aligned} a^3 \cdot a^4 &= (a \cdot a \cdot a)(a \cdot a \cdot a \cdot a) \\ &= a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \\ &= a^7. \end{aligned}$$

It appears, therefore, that the desired product can be obtained by adding the exponents. In general, $a^m a^n = a^{m+n}$.

PROBLEMS

Perform the indicated multiplications:

1. $x^5 \cdot x^3$,
2. $a^3 \cdot b^3 \cdot a$,
3. $(1 + r) \cdot (1 + r)$,
4. $x^2y \cdot x^3y^2$,
5. $2ab^2 \cdot 3a^2b$,
6. $100(1.04)^5 \cdot (1.04)^7$,
7. $(1.02)^3 \cdot (1.02)^2 \cdot (1.02)$,
8. $x^3y^6 \cdot x^7 \cdot x^9y^6$,
9. $abc \cdot a^3bc \cdot abc^2$,
10. $P(1 + r)(1 + r)^2$.

Illustration 2: $a^5 \div a^2 = ?$

The division to be performed may be written

$$\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a}$$

Canceling factors common to both numerator and denominator, we have

$$\frac{a^5}{a^2} = a \cdot a \cdot a = a^3.$$

From this it appears that the quotient can be obtained by subtracting the exponents. In general, $a^m/a^n = a^{m-n}$, if m is greater than n, and a is not zero.

Illustration 3: $a^5b^4 \div a^2b = ?$

We may write the indicated division as

$$\frac{a^5b^4}{a^2b} = \frac{a^5}{a^2}\frac{b^4}{b}.$$

Then applying the rule of Illustration 2 to each of the fractions in the right-hand member, we have

$$\frac{a^5b^4}{a^2b} = a^{5-2} \cdot b^{4-1} = a^3b^3.$$

PROBLEMS

Perform the indicated operations:

1. $x^6 \div x^4$,
2. $3^3 \div 3^2$,
3. $(1 + r)^3 \div (1 + r)$,
4. $x^3y^2 \div xy$,
5. $\frac{a^4b^5c^3}{a^2b^2c}$,
6. $\frac{2 \cdot 16^2}{4^2 \cdot 2^3}$,
7. $\frac{L^3R^7}{LR^2}$,
8. $\frac{P(1 + r)^6}{1 + r}$,
9. $\frac{2^3b^3}{3^2b}$.

Illustration 4: $(a^2)^3 = ?$

The a^2, which may be written $a \cdot a$, is to be taken three times as a factor. Thus $(a^2)^3 = (a \cdot a) \cdot (a \cdot a) \cdot (a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a = a^6 = a^{2\cdot 3}$.

It appears, therefore, that in general the rule in this case is

$$(a^m)^n = a^{mn}.$$

Illustration 5: $(abc)^3 = ?$

The (abc) is to be taken three times as a factor. Thus $(abc)^3 = (abc) \cdot (abc) \cdot (abc) = a \cdot b \cdot c \cdot a \cdot b \cdot c \cdot a \cdot b \cdot c = a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot c \cdot c \cdot c = a^3b^3c^3$. It appears, then, that each factor within the parentheses is raised to the power indicated. The general law is

$$(abc \cdots)^m = a^mb^mc^m \cdots.$$

PROBLEMS

Perform the indicated operations:

1. $(x^2)^2$,
2. $(ab)^2$,
3. $(x^2y^3)^3$,
4. $\frac{(xy^2)^3(x^2y)}{x^3y^3}$,

5. $\dfrac{(1.06)^3 \cdot (1.06^2)^5}{(1.06)^4}$,
6. $(abc)(abc)^2$,
7. $(2y)^3 \cdot 2y^3$,
8. $(3x^2y)^4 \cdot 3x^2y^4$,
9. $(ac)^2 \cdot ac^2$,
10. $[(1 + i)^2(1 + j)^3]^4$,
11. $(1.05)^3 \cdot (1.05^2)^4$,
12. $(xyz^2)^3 \cdot xyz^2$,
13. $(7P)^2 \cdot 7P^2$,
14. $(e^2m^3n)^2(em)^3(mn)^2$,
15. $[3(1 + r)]^3(1 + r)^2 \cdot r^3$,
16. $[(a + e)]^2(a + e) \cdot e^2$.

Illustration 6: $27 \cdot 9^3 \cdot 3^4 = ?$

Since $27 = 3^3$, and $9 = 3^2$, the expression may be written

$$3^3 \cdot (3^2)^3 \cdot 3^4 = 3^3 \cdot 3^6 \cdot 3^4 = 3^{13}.$$

PROBLEMS

Perform the following indicated operations, making use of the properties of exponents that have just been learned:

1. $\dfrac{2^3 \cdot 3^4}{2^2 \cdot 3}$,
2. $\dfrac{6x^{14}y^9}{2x^8y^7}$,
3. $x^4 \cdot x^5 \cdot x^6$,
4. $SR \cdot R \cdot R^2$,
5. $9ri^2 \cdot 6r^2i$,
6. $(3a^3b^5)(2ab^3)$,
7. $(2x^2)^3$,
8. $(x^4 \div x^3) \cdot x^5$,
9. $(x^5 \cdot x^2) \div x^4$,
10. $(a^3b^2)(a^2b^4)$,
11. $\dfrac{3}{y}\,(9y^2z)$,
12. $\dfrac{(0.5a^2i)(0.05ei^2)}{ei}$,
13. $(1 + r)^3(1 + r)^2$,
14. $\dfrac{(6PR)^2}{6PR^2}$,
15. $\dfrac{(Prt)^2r}{t}$,
16. $6^3 \cdot 2^2 \cdot 9^2$,
17. $2 \cdot 3^2$,
18. $3^2 \cdot 2^3$,
19. $16 \cdot 4^3 \cdot 8^2 \cdot 2$,
20. $(5ea) \cdot \dfrac{1}{5e}$,
21. $\frac{3}{4}uv(2^3u^2)^3$,
22. $\dfrac{(0.2rt^2)(0.02r^2t)}{rt}$,
23. $(a + e)^4(a + e)^2 \cdot ae$,
24. $\dfrac{(8wz)^3}{8wz^3}$,
25. $\dfrac{(Sdt)^3d}{t}$,
26. $15^3 \cdot 9^2 \cdot 4S$,
27. $16 \cdot 8^2 \cdot 4^3 \cdot 2$,
28. $\dfrac{x^3y^2z^5 \cdot (xy)^4}{xyz}$,

29. $\dfrac{(ab^3c)^2 \cdot abc^3}{a^2b^5c^4}$,

30. $(2mn) \cdot \dfrac{1}{2m}$,

31. $\dfrac{49^2(w^2z)^5w^2z}{(7^2w)^2z^6}$,

32. $\dfrac{(abc)^2 \cdot abc^2}{abc}$,

33. $(2pq)^4 \cdot \dfrac{1}{2p}$,

34. $(0.5sr^2)^2 \cdot \dfrac{1}{sr}$,

35. $(a + e)^3 \cdot \dfrac{1}{a + e}$,

36. $\dfrac{(2ASK)^3}{8} \cdot 4A$,

37. $\dfrac{x^4y^4}{x^4 + y^4} \cdot \dfrac{1}{xy}$,

38. $\dfrac{2^5}{3^5} \cdot \dfrac{3^6}{2^6} \cdot 2$,

39. $\dfrac{i^3 + j^3}{i^3j^3} \cdot (ij)^3$,

40. $\dfrac{m^4}{1 + m^4} \cdot \dfrac{1 + m^4}{m^3}$,

41. $\dfrac{(X^2)^3}{X^5} - 1$,

42. $\dfrac{(Y^3)^4}{Y^7} - 1$.

28. Multiplication. We have already learned that multiplication of positive whole numbers is a shortened process of addition and have seen that multiplication may be carried out when letters or other symbols represent quantities as well as in the case of numbers. But now that we are dealing with both positive and negative quantities, we must have some rules for the sign of a product when combinations of these numbers are multiplied. The establishment of these rules may best be approached through examples.

Examples:

1. $(+3)(+4) = ?$

Here we see that $+3$ is to be added four times:

$$+3 + 3 + 3 + 3 = +12,$$

or

$$(+3)(+4) = +12.$$

Likewise,

$$(+a)(+b) = +ab.$$

2. $(-3)(+4) = ?$

We are now to add -3 four times:

$$(-3) + (-3) + (-3) + (-3) = -12,$$

or

$$(-3)(+4) = -12.$$

Likewise,

$$(-a)(+b) = -ab.$$

3. $(+3)(-4) = ?$

In this example the $+3$ is to be subtracted four times:

$$-(+3) - (+3) - (+3) - (+3) = -12.$$

Note that we would have obtained the same result by adding the -4 three times.

4. $(-3)(-4) = ?$

Here -3 is to be subtracted four times:

$$-(-3) - (-3) - (-3) - (-3) = +3 + 3 + 3 + 3 = +12.$$

Likewise, $$(-a)(-b) = ab.$$

From the foregoing considerations we have the following rules:

1. *The product of two quantities of like signs is positive.*
2. *The product of two quantities of unlike signs is negative.*

When three or more numbers are multiplied we may apply the foregoing rules successively. The student may easily verify that the product of an even number of negative factors is positive, whereas the product of an odd number of negative factors is negative.

The product of a polynomial and a monomial is the *algebraic sum* of the products obtained by multiplying each term of the polynomial by the monomial. Thus, by the distributive law (Section 22) we write

$$a(a + b - c) = a^2 + ab - ac.$$

Problems

Find the product of the following:

1. $(-12)(-12)$,
2. $(6)(-3)$,
3. $(-4)(16)$,
4. $(-\frac{1}{2})(-\frac{2}{3})$,
5. $(\frac{3}{4})(-2\frac{1}{8})$,
6. $(-99)(-100)$,
7. $(15)(-2\frac{1}{5})$,
8. $(-3\frac{1}{4})(-2\frac{7}{8})$,
9. $(-a)(2a)$,
10. $(3b)(-5a)$,
11. $2c \cdot 7c$,
12. $(4R)(-3R)$,
13. $9z^2x \cdot 4z^2$,
14. $8y^3x^2 \cdot (-\frac{1}{2}yx)$,
15. $R \cdot (PRt)$,
16. $(-3a^3b^5)(-2ab^3)$,
17. $\frac{6}{5}a^3y \cdot (-\frac{25}{3}a)$,
18. $15 \cdot (-\frac{1}{3}abc^3)$,

19. $\frac{3}{y} \cdot (27y^2)$,
20. $(0.4e^2i)(2ix)(-6exy)$,
21. $2(x + y)$,
22. $3(a + b)$,
23. $2a(a - b)$,
24. $3x(x + 1)$,
25. $-2(x + y - 3)$,
26. $\left(-\frac{3}{a}\right)\left(a - \frac{a}{3} + \frac{1}{3}\right)$,
27. $x(a + c - 4)$,
28. $x(5 - c + d)$,
29. $d(P - Pd)$,
30. $P(1 + rt)$,
31. $A(1 - dt)$,
32. $a(1 - r^n)$,
33. $x^2(1 - x^3)$,
34. $y^3(y - 2)$,
35. $-xy(x + y) + 3(x^2y + xy^2)$,
36. $cd^2(c^2 - d^2) - d^2(c^3 + 1)$,
37. $\left(\frac{p^3}{q^4}\right)(q^8 + p) - (p^3q^4 + p - 1)$,
38. $-x^3y^4\left(\frac{1}{(xy)^2} - xy^2\right) + 2xy^2(1 - x^3y^4)$.

29. Factoring. By actual multiplication, we have found that

$$a(a + b - c) = a^2 + ab - ac.$$

The right-hand member of this particular example is in *expanded* form, and the left-hand member is in *factor* form. The process of finding the simple factors of an expression given in expanded form is called *factoring*. There are many types that may be recognized from the expanded form, but perhaps the most common one is the product of a monomial and a polynomial. Monomial factors can usually be detected by inspection.

Illustration: Factor $6a^2bc - 12a^3b^5$.
Solution: We see that $6a^2b$ is common to both terms. Hence,

$$6a^2bc - 12a^3b^5 = 6a^2b(c - 2ab^4).$$

Check: By multiplying the factors we get $6a^2bc - 12a^3b^5$.

PROBLEMS

Factor the following algebraic expressions, and check by multiplication:

1. $a + ab$,
2. $P + Prt$,
3. $3a - 12b + 9c$,
4. $2R + 4S + 16T$,
5. $4ax - 8ax^2$,
6. $3a^2b + 9ab^2$,
7. $3x^2y^3z^5 - 4xy^2$,
8. $7abc^4 + 28b^5c$,
9. $2MN - 8M^2N + 20MN^3$,
10. $P^2R + 5PR^5 - 7P^3R$,
11. $(2x^2y)^2(xy)^3 - 12x^5y^2$,
12. $PQ(2PQ^2)^3 + 4PQ^3$,
13. $(0.2rt^3)^2 - 0.04r^3t^5$,
14. $(2.4uv)^2 + 2.4u^4v$,
15. $ax + bx$,
16. $cx + 2x$,
17. $P - Prt$,
18. $A - Adt$.

30. Division. In our treatment of the multiplication of positive and negative numbers we arrived at certain rules for the sign of a product by considering four different examples. Since division is the inverse of multiplication, we may conveniently use these same examples in establishing corresponding rules for division.

Examples:

1. $(+12) \div (+4) = +3$ because $(+3) \cdot (+4) = +12$.
2. $(-12) \div (+4) = -3$ because $(-3) \cdot (+4) = -12$.
3. $(-12) \div (-4) = +3$ because $(+3) \cdot (-4) = -12$.
4. $(+12) \div (-4) = -3$ because $(-3) \cdot (-4) = +12$.

It is apparent that the following rules hold for division:

1. *The quotient of two quantities of like signs is positive.*
2. *The quotient of two quantities of unlike signs is negative.*

PROBLEMS

Find the quotient in each of the following:

1. $(-4a) \div (a)$,
2. $3xy \div 3x$,
3. $(-12r) \div 9$,
4. $(-9x^2z) \div (2xz)$,
5. $(6v^3w) \div (-4vw)$,
6. $\frac{6}{4}a^3y \div \frac{1}{2}ya$,
7. $18abc^4 \div 3ac^2$,
8. $(-0.04e^2) \div (-e)$,
9. $(-31m^5n^3) \div (9m^2n)$,
10. $ar^6 \div r$,
11. $(1.08)^{10} \div (1.08)^6$,
12. $(-4x)(3y) \div (3x)(2y)$,
13. $(-5a)(-3b)(2c) \div (2a)(-3c)$,
14. $(400)Prt \div (-200)r$,
15. $(2.046D)(1.2) \div (-0.40D)$,
16. $(42a^5R^7S^5) \div (-7aR^4S)(RS)$,
17. $(-2xy)^2 \div [-(2xy)^2]$.

31. Simple Equations. An estate of \$11,000 was divided so that the daughter received $\frac{1}{2}$ as much as the widow and the son received $\frac{2}{3}$ as much as the daughter. How much did each receive?

The problem can be formulated mathematically and expressed very simply if we resort to the language of algebra and state it in the form of an *equation.* In other words, our first task is much like that of making a translation from one language to another. The problem is expressed in words. If we express the problem in algebra, we may take advantage of the simplicity which operations with symbols afford. This means that we must first of all understand the problem and know definitely what is called for. After setting up an equality involving the known quantities and a symbol standing for the quantity we seek, we are in a position to solve for that unknown and continue with a full solution. We proceed as follows: Let

$$x = \text{widow's share.}$$

Then,

$$\frac{x}{2} = \text{daughter's share,}$$

and

$$\frac{2}{3}\left(\frac{1}{2}x\right) = \frac{x}{3} = \text{son's share.}$$

The problem states that the three shares together total \$11,000. Hence, we write the equation

$$x + \frac{x}{2} + \frac{x}{3} = \$11{,}000.$$

After setting up the equation, we are naturally faced with the second aspect of the solution of the problem. Our first task is to find a *solution of the equation,* that is, the value, or values, of the unknown for which the equation is true. This task, of course, will depend upon the type of equation that confronts us. Sometimes, the solution will be quite easy to find, and at other times it will require skillful algebraic manipulation. In the case at hand, the three terms on the left in our equation may be added together and written as $\frac{11}{6}x$, so that the equation may be written

$$\tfrac{11}{6}x = \$11{,}000.$$

Dividing both sides by $\frac{11}{6}$, we find

$$x = \$\ 6{,}000 \qquad \text{(widow's share).}$$

Thus $\frac{x}{2} = \$\ 3{,}000$ (daughter's share),

and $\frac{x}{3} = \$\ 2{,}000$ (son's share).

Check:

$$\$6{,}000 + \frac{\$6{,}000}{2} + \frac{\$6{,}000}{3} = \$6{,}000 + \$3{,}000 + \$2{,}000 = \$11{,}000.$$

The equation in the above problem in which the unknown is of the first power is said to be an equation of the first degree (linear) in x; and $x = \$6{,}000$ is called a solution of the equation. Other examples of first-degree equations in one unknown are:

(1) $4x + 3x = 18 - 2x,$

(2) $P - 40 = 120,$

(3) $3(2x - 5) = 12x - 9,$

(4) $\frac{x-1}{x-3} = -2,$

(5) $10S = \$1{,}150 - S,$

(6) $\frac{x}{2} = \frac{3}{4}(124 + x).$

We shall now solve a few of these examples and at the same time point out certain steps that aid in their solutions.

1. $4x + 3x = 18 - 2x.$

Adding $2x$ to each side, $4x + 3x + 2x = 18 - 2x + 2x.$

Collecting terms, $9x = 18.$

Dividing by the coefficient of x, $x = 2.$

We observe that the operation of adding $2x$ to both members of the equation in this example is equivalent to removing $-2x$ from the right member and writing $+2x$ in the left member. This process is often referred to as "transposing." In carrying out the steps in the solution of an equation, we must at all times observe the fundamental assumptions of algebra (see Section 22). That $x = 2$ is a solution may be verified by substitution:

$$4(2) + 3(2) = 18 - 2(2)$$
$$14 = 14$$

2. $P - 40 = 120.$

Transposing and changing the sign, $P = 120 + 40.$

Collecting terms, $P = 160.$

Check: $160 - 40 = 120.$

3. $3(2x - 5) = 12x - 9.$

Removing parentheses, $6x - 15 = 12x - 9.$

Transposing, $6x - 12x = -9 + 15.$

Collecting terms, $-6x = 6.$

Dividing by the coefficient of x, $x = -1.$

Check:

$$3[2(-1) - 5] = 3(-2 - 5) = 3(-7) = -21 = 12(-1) - 9.$$

4. $\dfrac{x - 1}{x - 3} = -2.$

Clearing of fractions, by multiplying both members by $x - 3$,

$$x - 1 = -2(x - 3).$$

Removing parentheses, $x - 1 = -2x + 6.$

Transposing, $x + 2x = 6 + 1.$

Collecting terms, $3x = 7.$

Dividing by 3, $x = \frac{7}{3}.$

Check: $\dfrac{\frac{7}{3} - 1}{\frac{7}{3} - 3} = \dfrac{\frac{4}{3}}{-\frac{2}{3}} = \dfrac{4}{3}\left(-\dfrac{3}{2}\right) = -2.$

The student should solve examples (5) and (6) (page 69) and verify the solutions $S = \$104.55$ and $x = -372$, respectively.

The steps involved in solving first-degree equations in one unknown may be summarized as follows:

1. Clear of fractions, and remove any parentheses containing the unknown.
2. Collect terms containing the unknown on one side of the equation, and collect all other terms on the other side.
3. Obtain the coefficient of the unknown by factoring out the unknown.
4. Divide through by the coefficient of the unknown.
5. Check the result by substitution in the original equation.

PROBLEMS

1. In the following equations solve for the unknown, and check the result by substituting into the original equation:

(*a*) $10x - 6 = 3 - 4$,
(*b*) $2x - 2 = 16$,
(*c*) $15 - 13m = 21 - 11m$,
(*d*) $8b - 3(2b - 3) = 1 - 9(2b + 7)$,
(*e*) $-5 - 2(3y - 2) = 7y - 4(7y - 6)$,
(*f*) $1 - P = P + 3(1 - P)$,
(*g*) $y - \frac{3}{4} = \frac{1}{4} - y$,
(*h*) $\frac{1}{3}x - \frac{1}{2} = x + \frac{1}{4}$,
(*i*) $\frac{x}{5} - 5 = \frac{x}{10} + 5$,
(*j*) $-x = -4 + 2$.

2. Solve the following equations:

(*a*) $\frac{3}{8}x = \frac{1}{8}(17 - x)$,
(*b*) $\frac{1 - \frac{1}{x}}{5} = 2$,
(*c*) $1.05 - 3x = \frac{2}{7}(2x - 7)$,
(*d*) $\frac{r - 2}{r + 3} = \frac{4}{9}$,
(*e*) $\frac{S}{1 + 0.02} = 153$,
(*f*) $\frac{1}{2} = \frac{1}{1 - r}$,
(*g*) $6 = (100)(\frac{1}{2})r$,
(*h*) $18 = 2 + (n - 1)2$,
(*i*) $200 = \frac{F}{(1 + 0.03)^2}$,
(*j*) $100 = \frac{9}{2}(1 + L)$,
(*k*) $S - 2S = 3 - 3 \cdot 2^4$,
(*l*) $x - (2x + 1) = -1$,
(*m*) $60 = \frac{5}{2}(2 + L)$,
(*n*) $\frac{1}{2}x + 2(x - 1) = \frac{1}{2}$,
(*o*) $\frac{3}{4} = \frac{1}{x} - 1$.

3. Solve $ax - x = \frac{a - 1}{2}$ for x; for a.

4. Solve $D = Adt$ for A; for d; for t.

5. Solve $S = \frac{a - ar^n}{1 - r}$ for a.

6. Solve $C = 2\pi r$ for r.

7. Solve $A = \frac{1}{2}bh$ for h; for b.

8. Solve $a(b - 1) = 2a + b$ for b; for a.

9. Solve for A, and then find the value of A when $P = 100$ and $r = 0.04$.

$$P = \frac{A}{(1 + r)^2}.$$

10. Solve $I = Prt$ for r, and find r if $I = 10$, $P = 500$, and $t = \frac{1}{2}$.

11. Solve $A = P + Prt$ for r, and find r if $A = 309$, $P = 300$, and $t = \frac{2}{3}$.

12. Solve $P = A - Adt$ for t, and find t if $P = 882$, $A = 900$, and $d = 0.04$.

13. A hat is sold for $10, yielding the seller a profit of $\frac{1}{3}$ of the cost. Let C stand for cost, and write the appropriate equation that expresses the relation Cost plus Profit equals Selling price. Solve this equation for C, thus obtaining the cost at which the hat had been originally purchased.

14. A can of beans cost a grocer 12¢. If *overhead* on the can is taken as $\frac{1}{100}$ of its selling price, for what amount should the can be sold to realize a *net profit* of $\frac{3}{100}$ of the selling price?

15. An expenditure of $1,000 is to be apportioned between two divisions of a factory. Division A is to be charged 7 times the amount charged to Division B. How much should be charged to each division?

16. A man pays $200 property tax on his cottage and lot. If the tax rate is $3.20 per $100 of the assessed value of the property, what is the assessed value?

17. The Adams Steel Company wished to construct a fence around its property. The plot of ground was rectangular in shape and was three times as long as it was wide. Of 1,091 rods of wire purchased for the fencing, 11 rods were not needed. What were the dimensions of the property?

18. The manager of a small industrial firm received one-tenth of the net profits of the firm as a salary, after deducting the amount of his salary as an expense to the firm. What was his salary, if profits for the year were $35,000?

32. Ratio. In business practice we are often called upon to state the relationship between two quantities. For example, suppose there are 120 men working in a certain department while 40 men are working in another department. How can we express the relative magnitude of the two groups of men? Evidently, there are *three* times as many men in the larger department as in the smaller department. We obtain the index 3 by dividing 120 by 40 and say that the ratio of 120 to 40 is 3, which is sometimes read "3 to 1."

The single number that expresses the value of such a relation ship is called the *ratio* of the two numbers involved. This ratio is seen to be the quotient of the first number divided by the second. Thus, $\frac{120}{40} = 3$ is called the "ratio of 120 to 40." For another department having 35 men, the ratio would be written $\frac{120}{35}$, or $\frac{24}{7}$.

A ratio between two numbers can always be expressed as a fraction. Sometimes the sign (:) is used to indicate a ratio, but the meaning is the same no matter how the division is indicated. Thus, the fraction $\frac{120}{40}$, the division $120 \div 40$, and the expression 120:40 all express the same thing and are read "the ratio of 120 to 40." The ratio 40:120 is the *inverse* of the ratio 120:40. It should be emphasized that the ratio of two quantities can be found only when they are expressed in the same units. Thus, the ratio of 12 feet to 8 inches cannot be computed until 8 inches is changed to feet *or* 12 feet is changed to inches. Writing 8 inches as $\frac{2}{3}$ feet, we have for the ratio $12:\frac{2}{3}$, or 18.

Illustrations: Find the value of the following ratios:

1. 18 to 2. *Ans.* $\frac{18}{2} = 9$. "The ratio of 18 to 2 is 9."
2. 7 to 3. *Ans.* $\frac{7}{3} = 2\frac{1}{3}$. "The ratio of 7 to 3 is $2\frac{1}{3}$."
3. 15 to 4. *Ans.* $\frac{15}{4} = 3.75$. "The ratio of 15 to 4 is 3.75."
4. 48 hr to 8 hr. *Ans.* $\frac{48}{8} = 6$. "The ratio of 48 hours to 8 hours is 6."
5. $\frac{3}{5}$ to $\frac{3}{4}$. *Ans.* $\frac{3}{5} \div \frac{3}{4} = \frac{3}{5} \cdot \frac{4}{3} = \frac{4}{5}$.

"$\frac{3}{5}$ and $\frac{3}{4}$ are in the ratio of 4 to 5."

6. 5 hr to 30 min. *Ans.* 5 hr to $\frac{1}{2}$ hr $= \dfrac{5}{\frac{1}{2}} = 10$. "The ratio of 5 hr to 30 min is 10."

PROBLEMS

1. Find the values of the following ratios: 10 to 5, 15 to 7, 19 to $2\frac{1}{2}$, $\frac{5}{9}$ to $\frac{10}{12}$, \$64 to \$16, 11 oz to 3 oz, 52 dozen to 42 dozen, 6 hr to 40 min, 8 yd to 2 ft.

2. Write the inverses of the following ratios: 8 ft to 3 ft, 12 to 6, 1.5 to 0.5, 44 hr to 12 hr, \$4 to \$0.20, 10¢ to \$10, 50 sec to 1 min.

3. The output of a certain motor is 3,200 watts, and the input is 4,000 watts. What is the ratio of the output to the input?

4. In a certain telephone exchange it was found that between 6 P.M. and 7 P.M. there were 356 local calls and 12 long-distance calls. Night rates went into effect at 7 P.M., and during the next hour there were 212 local calls and 53 long-distance calls. What was the ratio of local to long-distance calls for each of the two hours?

5. Divide \$25 between *A* and *B* in the ratio of 2 to 3.

6. For selling \$1,500, \$2,000, \$2,500, and \$3,000 above their quotas, four salesmen received a total bonus of \$900. How much did each receive?

7. In a certain college there are 3,200 students and 294 instructors. What is the ratio of the number of students to the number of instructors?

8. In making a survey of radio listeners, 3,296 replies were received from 10,000 questionnaires sent out. What is the ratio of replies to the total number of questionnaires? How many replies per 100 questionnaires were received?

9. The Big Wheel Manufacturing Company sells bicycles for \$50 each. Raw materials cost \$12, labor \$25, and overhead \$3. Find the ratio of the profit to the selling price, and also of each expense to the selling price.

10. The current assets of the Ajax Company are \$235,000, and its liabilities are \$94,000. Find the *current ratio*, that is, the ratio of assets to liabilities.

11. A shoe store had total sales of \$100,000 for the year, and its average inventory was \$32,000. Find the *merchandise turnover*, that is, the ratio of total sales to average inventory.

33. Proportion. A proportion is a statement of the equality of two ratios. Thus, the ratio of 120 to 40 is 3, $\frac{120}{40} = 3$; also the ratio 15 to 5 is 3, $\frac{15}{5} = 3$. Therefore, the two ratios are equal and the numbers 120, 40, 15, and 5 are said to be in proportion. We may write the proportion in any of the following ways:

$$120:40 :: 15:5,$$

$$120:40 = 15:5,$$

$$\tfrac{120}{40} = \tfrac{15}{5}.$$

All three of these statements mean the same thing and may be read "120 is to 40 as 15 is to 5."

The fractional form of writing a proportion is, perhaps, most useful because it is expressed in the language of algebra and confines our attention to a very simple equation. It is a simple form of fractional equation, the solution of which may be found by techniques for solving such equations.

Illustrations:

1. In the proportion $3:5 = 6:x$, find the unknown term.

The given proportion may be written

$$\frac{3}{5} = \frac{6}{x}.$$

Cross multiply, $3x = 30$,

and $x = 10$.

2. *A* loans $800 and receives an annual income of $52. *B* loans a sum at the same rate and receives an annual income of $91. How much does *B* loan?

We may write the proportion

Large Income:Small Income = Large Loan:Small Loan.

We shall then have

$$91:52 = x:800,$$

where x is the missing term. By writing the proportion in the fractional form

$$\frac{91}{52} = \frac{x}{800},$$

we obtain the result simply by solving for x.

Cross multiply, $52x = 72{,}800,$

and $x = 1{,}400.$

B therefore loans $1,400.

3. If 6 men can hoe a field of corn in 3 days, how long will it take 4 men to hoe the field?

Since 4 men will require more time, a correct proportion would be

Large Group : Small Group = Large Time : Small Time.

That is, $\frac{6}{4} = \frac{x}{3}.$

Hence, $4x = 18,$

and $x = 4\frac{1}{2}$, number of days required of the 4 men.

PROBLEMS

1. Solve for x:

(a) $\frac{x}{7} = \frac{4}{2}$

(b) $\frac{x+1}{x-2} = \frac{6}{2},$

(c) $\frac{8}{3x-5} = \frac{4}{x+6},$

(d) $\frac{2x}{1.2} = \frac{5}{0.6}.$

2. If 12 lb of bacon costs $9, how much will 20 lb cost?

3. If 25 men can do a piece of work in 10 days, how long will it take 20 men to do the same work?

4. An investment produces an income of $600 at $3\frac{1}{2}$ per cent. What would it produce at 5 per cent?

5. The taxes on a house assessed for $15,000 are $290. How much should the taxes be on a house assessed for $8,500?

6. If 1 in. on a map represents 75 miles, how many inches will represent 660 miles?

7. If a carpet strip 9 ft long costs $37.50, how much will 23 ft of the same material cost?

8. Two partners divide a profit of $1,500 in the ratio of 3 to 7. What does each receive?

9. An apprentice received \$207 for 30 days' pay. The next month he worked 26 days at the same rate and had \$2.70 deducted for insurance. How much was his take-home pay?

10. A train travels 380 miles in 10 hr. How far at the same rate will it travel in 16 hr 30 min?

11. Divide \$200 in the ratio 2 to 3.

12. Divide \$350 in the ratio 3 to 4.

13. Jones, Jenkins, and Johnson receive \$20,000 profit from their partnership for the year. How much does each receive if Jenkins and Jones each receive half as much as Johnson?

14. A rectangular display is 10 ft long and w ft wide. How wide should it be made if it is desired to have it only $\frac{2}{3}$ as long with the same display area?

15. If only 200 employees of a factory are given a raise, each will receive \$5 more per week. What will be the amount of the raise for each employee if 250 employees share the same total increase?

16. A factory floor can be swept in 2 hr by 2 men. At the same rate, how long should it take 5 men to sweep the floor?

Exercises

1. Add:

(*a*) $4a,\ -3a,\ 9a,$
(*b*) $-7ax,\ 5ax,\ 12ax,$
(*c*) $3a + 5b,\ -2a - b,\ 4a - 2b$
(*d*) $r(x + y),\ s(x + y),\ -ry,$
(*e*) $a^2 - 2ab + b^2,\ a^2 + 4ab - 3b^2$

2. Subtract:

(*a*) $5a$ from $12a,$
(*b*) $72x$ from $15ax,$
(*c*) $-3ay$ from $5ay,$
(*d*) $a^2 - 3ab + b^2$ from $ab - 2b^2$
(*e*) The sum of $(p + q + r)$ and $(-2p + 2q - 3r)$ from the sum of $(3p - q + r)$ and $(q - 2r)$.

3. Remove signs of aggregation, and simplify:

(*a*) $(a + b - 1) + (3a + 2b - 4c),$
(*b*) $(a + 2b - 3c) - (2a - 4b - 3c)$
(*c*) $(1 + R) + (-2 + S + 4R) - (7S - 2R)$

(*d*) $a^2 - [a + (3a^2 - 5) - 2(a + 4)]$
(*e*) $(x - y + 4) - \{3x - [2y + 5 + (x - 3y)] - 3(x + y)\}$

4. Multiply:

(*a*) $3a$ by $5ab$,
(*b*) $3ab$ by $-2ab$
(*c*) Prt by $2K$,
(*d*) $3x^2y$ by $-3xy^2$,
(*e*) $2CD_x$ by $4D_x$.

5. Divide:

(*a*) $6ab$ by $3a$,
(*b*) $(1.015)^{12}$ by $(1.015)^4$
(*c*) $3xyz$ by axy,
(*d*) $48Rp^3$ by Rp,
(*e*) $1.02MN$ by $0.2M$

6. Solve for x:

(*a*) $3x + 4 = 7x - 12$,
(*b*) $2(x - 7) = 3(x + 2)$,
(*c*) $5(x + 1) = 3(x - \frac{1}{3}) + 2$,
(*d*) $3(x - b) = 24$,
(*e*) $2(9x - 2) - 4(8x + 9) = -3(x + 6)$.

7. Solve for P, n, and i: $I = Pni$. Find P if $I = 1.4$, $i = 0.04$, and $n = \frac{1}{4}$.

8. Solve for P, n, and i: $S = P(1 + ni)$. Find P if $S = 222.5$, $i = 0.05$, and $n = \frac{1}{2}$.

9. Solve for P: $S = P(1 + i)^n$. Find P if $S = 1{,}060.9$, $i = 0.03$, and $n = 2$.

10. Solve for a, d, and n: $L = a + (n - 1)d$.

11. Solve for a, n, and L: $S = \frac{n}{2}(a + L)$.

12. Solve for C_1 and C_2: $\frac{1}{2}(4 - 2C_1) = C_2(C_1 - 2) + 1$.

13. A man is x years old. Express in algebraic statements: (*a*) his age 5 years ago, (*b*) his age 12 years from now, (*c*) one-half his age N years from now, (*d*) how long before he becomes 35 years old, (*e*) how long since he was 20 years old.

14. Three articles cost \$52. The second cost \$4 more than the first, and the third cost twice as much as the first. Find the cost of each article.

15. A merchant has 50 lb of coffee, selling at \$1.00 per pound. How many pounds of \$1.15 coffee must he add to the 50 lb to make a mixture to sell at \$1.12 per pound?

16. An employer pays \$25 a day for skilled labor and \$15 a day for unskilled labor. The daily pay roll is \$515 for 25 men. How many of each kind of laborer are there?

17. What part of a piece of work can Smith do in 1 day if he can do all of the work (*a*) in 6 days, (*b*) in 13 days, (*c*) in $2\frac{1}{2}$ days, (*d*) in K days, (*e*) in $m + n$ days?

18. If $16\frac{3}{4}$ yd of material costs \$18, how much will 100 yd cost?

19. The dimensions of a rug are proportional to the floor dimensions of the room in which it is used. If the room is 15 by 18 ft, what is the width of a rug if it is to be 14 ft long?

20. In a school annual the editor wants a photograph having dimensions 9 by 7 in. to be made into a copper engraving 4 by $3\frac{1}{2}$ in. Can this be done?

21. Jones cleans the walls of a room of dimensions 18 by 22 by 11 ft for \$3.63. How much should he charge, at the same rate, to clean the walls of a room of dimensions 25 by 20 by 12 ft?

22. In bronze, 11 parts of tin combine with 39 parts of copper. How many pounds of tin will combine with 312 pounds of copper?

23. How many men must a contractor add to his pay roll in order to finish a job in 6 days if he has already figured 150 men could do the job in 9 days?

24. A map is drawn using the scale 1 in. = 500 miles. What length on the map should be used to denote 3,450 miles? To denote 55 miles?

Self-Test

Time: 50 Minutes

(Score 5 points for each problem correctly solved)

1. An airplane travels q miles in t hr. How far does it travel in $2H$ min?

2. Simplify by removing all symbols of grouping:

$$7x - [3x + 4 + 5x - (2x - 3)]$$

3. The profit on a certain article was found to equal the difference between the selling price and the sum of the cost and expenses. Use single-letter symbols, and write an equation for the cost.

Perform the indicated operations, and simplify results:

4. $\dfrac{13r}{5} - 2r - \dfrac{2r}{3}.$

5. $\dfrac{Prt}{150 + r} \div \dfrac{P}{150}.$

6. $(2x^2)^3 \div (4x)(4x)^2.$

7. $0.06z^2w(0.2w^2) - (0.04)(0.12zw^3).$

8. Solve for d: $(\frac{1}{2}d - 4) = \frac{1}{2}(3d - 16).$

9. What is the average selling price if m articles are sold at p cents each, b at q cents each, and c at r cents each?

10. A well-known bond price formula is written as follows:

$$P = K + \frac{g}{j}(C - K).$$

Solve for K.

11. In annuity problems the following formula is found in connection with present value. Solve for D_x.

$$a_x = \frac{D_x + D_{x+1} + D_{x+2} + D_{x+3}}{D_x}.$$

12. The amount of $1 at compound interest may be calculated from the formula $A = (1 + i)^n$. Find the amount A if $i = 0.04$ and $n = 6$.

13. The Bismuth Company's balance sheet for Dec. 31 shows current assets, $3,462,724.17, and current liabilities, $1,564,235.29. What is the ratio of assets to liabilities? (Round off to two decimal places.)

14. The weight of 12 ft of $\frac{1}{2}$-in. iron pipe is 14.6 lb. What is the weight of 420 ft of the same pipe?

15. How many pounds of cashew nuts, worth $1.25 a pound, must be mixed with 25 lb of peanuts, worth $0.60 a pound, to obtain a mixture worth $1 a pound?

16. Evaluate:

(*a*) $3(x - y)$ when $x = 2$, $y = 3$,
(*b*) x^2y^3 when $x = 3$, $y = 2$,
(*c*) $2a^3x^2(a - b)$ when $a = 12$, $x = 2$, $b = 8$,
(*d*) $\frac{1}{x} + \frac{5}{n} + \frac{3}{R}$ when $x = 3$, $n = 8$, $R = 12$.

17. Divide $93 between A and B so that A may receive $23 less than B.

18. A map is drawn using the scale 1 in. = 2 miles. What area on the map corresponds to 9 sq miles?

19. A certain street is paved, and assessments totaling $2,090.30 are made. The lots along the street have frontages of 30 ft, 20 ft, 60 ft, 75 ft, and 50 ft. What is the amount of the largest individual assessment?

20. Jones, Miller, and Adams together invest $300,000 in a business venture which returns 6 per cent profit in a given year. How much does each get if the profits are divided in the ratio 2 to 5 to 8?

Chapter III

INTRODUCTION TO STATISTICS

Introduction. In business and industry, answers to many questions are found through study and analysis of collected data. For example, from records, management determines wage averages, trends in accident rates, forecasts of future sales, effectiveness of promotional programs, and so forth. These are but a few of the many business situations where numerical information is at the basis of facts, trends, and relationships.

Tables, graphs, charts, and other devices are often useful in presenting information or in describing certain features of collected data. At other times, it is convenient to use numerical quantities computed from the data, such as the average, the range, the scatter, the coefficient of correlation, and so forth. Recent developments in methods of sampling information also provide valuable tools for drawing dependable conclusions in regard to many business operations.

Statistics may be thought of as that branch of mathematics which deals with the numerical description of collected data and the drawing of conclusions from these data. A knowledge of statistics is particularly valuable when one deals with data which do not lend themselves to consideration as individual items. Through the use of statistics, business analyses can be made clearly, concisely, and economically.

34. Presentation of Data. Before statistical analyses can be made, the data must be gathered and arranged in some orderly fashion. The usual procedure is to arrange the numbers or items to be used in tables and to supplement the tabular presentation with appropriate graphs and charts.

There are no fixed rules for making tables, but they should be designed so as to present the material in an unambiguous and convenient form so that further classifications can readily be made. Ordinarily, the detailed characteristics of tables will depend upon the kind and the number of data collected and upon the nature of the statistical study to be made.

Table 1 summarizes the number of errors made by 20 students who took Form A and Form B of a certain typewriting speed test.

TABLE 1

TYPEWRITING SPEED TEST ERRORS

Student	*Number of Errors*		*Student*	*Number of Errors*	
	Form A	*Form B*		*Form A*	*Form B*
(1)	(2)	(3)	(1)	(2)	(3)
A	7	9	K	23	17
B	9	10	L	23	17
C	10	10	M	16	15
D	11	11	N	15	14
E	13	12	O	16	15
F	16	13	P	17	15
G	18	14	Q	20	16
H	19	15	R	19	17
I	21	16	S	17	18
J	23	17	T	20	18
				333	289

In this table, the data are arranged according to students and show the number of errors made by each student on each form of the test. Other examples of tables of a type familiar to the reader are on this and the following page.

BANK DEPOSITS IN THE UNITED STATES

Year	*Bank Deposits, trillions*	*Year*	*Bank Deposits, trillions*
1941	$78	1947	$153
1942	83	1948	156
1943	107	1949	156
1944	129	1950	164
1945	151	1951	172
1946	159	1952	184

Business Record of a Small Firm
(Fifth Year of Operation)

Month	*Gross Earnings*	*Expense*	*Profit*
Jan.	$16,000	$12,300	$3,700
Feb.	12,000	9,600	2,100
Mar.	17,000	11,800	5,200
Apr.	21,400	16,700	4,700
May	19,800	15,200	4,600
June	10,200	6,100	4,100
July	8,400	5,300	3,100
Aug.	8,100	5,100	3,000
Sept.	9,200	6,000	3,200
Oct.	13,500	9,600	4,900
Nov.	18,700	14,300	4,400
Dec.	17,600	14,200	3,400

The interpretation of data is often facilitated by the use of a *graph*. For example, consider the table of bank deposits for 1941–1952. We observe that the deposits are dependent upon the year, or time. Hence, for these data the time may be called the *independent quantity* and the amount of bank deposits the *dependent quantity*. A graph indicating the relationship between these two quantities may be constructed as follows:

Select a convenient unit of measure, and make a mark for each year along a horizontal line, beginning at the point of intersection of this line with a vertical line (Figure 1). On the vertical line mark off the bank deposits, letting each unit represent 10 trillion dollars. (The choice of the unit of measure is arbitrary, but it should conveniently represent the largest and the smallest values for each of the quantities. It may or may not be the same for each quantity.) Each of these units may then be further divided, if desired, for convenience of reading. With reference to these scaled lines each pair of numbers in the table locates a point that we may now plot. Thus, the number pair (1941, 78), entries in the first line of the table, is plotted as a point by moving 1 unit to the right on the base line and then upward 78 units. After all points have been plotted, the graph is drawn by connecting them with line segments as shown in Figure 1. A smooth curve may be sketched through the plotted points if there is reason to believe that the amount of bank deposits varies regularly throughout each year, and meaning can be attached to the lengths of vertical line segments drawn to the curve from every point along the time scale.

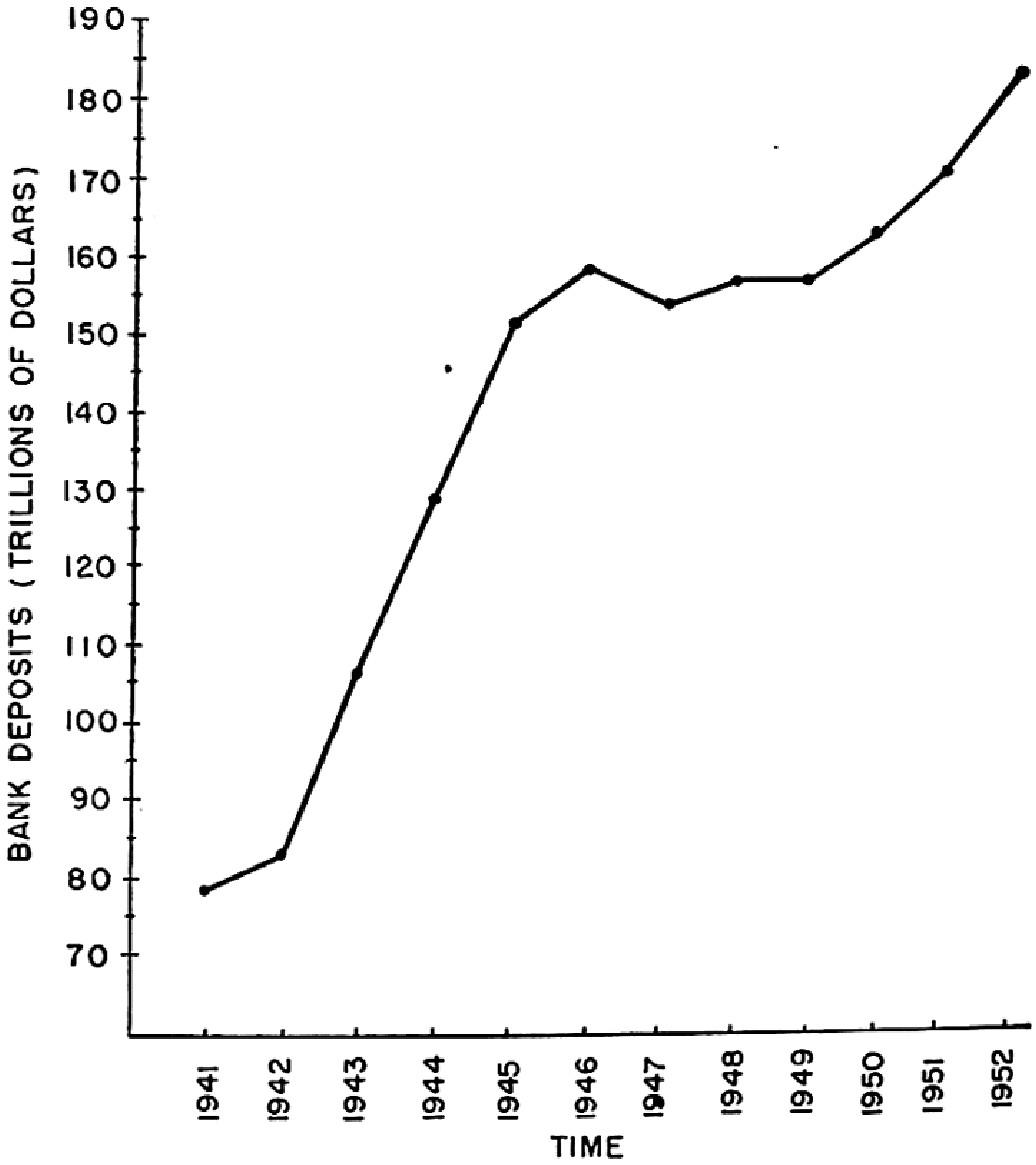


Figure 1. Bank Deposits in the United States

Pictorial bar charts, component area charts, pie charts, and maps are examples of other useful visual means of presenting data and are common in newspapers, magazines, and business journals.

Problems

1. Discuss the trend in bank deposits in the United States during the years 1941 through 1952. Draw a smooth curve through the plotted points in Figure 1, and estimate the amount of deposits on July 1, 1942. How would you explain the "flattening" of the curve between 1946 and 1949?

2. Refer to the small-firm business record shown on page 83, and graph the trend of business for the various months of the year. (Measure gross earnings, expense, and profit along the vertical scale, and obtain three curves.)

3. Depict the following data graphically:

NET RATE OF INTEREST EARNED ON INVESTED FUNDS*
(U.S. Life Insurance Companies)

Year	*Rate*	*Year*	*Rate*
1917	4.8%	1937	3.7%
1922	5.1	1942	3.4
1927	5.0	1947	2.9
1932	4.6	1952	3.3

* Source: *Life Insurance Fact Book*, 1953.

Give possible reasons for the rising and falling trends in the net rate of interest earned on invested funds.

4. Draw a line graph for the data given in the following table:

NUMBER OF FAILURES IN THE RETAIL TRADE*
(Monthly Averages)

Year	*Number*	*Year*	*Number*
1940	708	1947	102
1941	632	1948	182
1942	491	1949	354
1943	147	1950	369
1944	41	1951	341
1945	24	1952	319
1946	25		

* Source: *Business Statistics*, U.S. Department of Commerce, 1953.

Give an interpretation of the high and the low points on the graph.

5. The following data give a percentage distribution of assets of United States life insurance companies over a period of 13 years. Graph the data on the same chart, making three line graphs.

ASSETS OF UNITED STATES LIFE INSURANCE COMPANIES*

Year	*U.S. Government Securities*	*Securities of Business and Industry*	*Real Estate*
1940	19.3%	30.1%	6.7%
1941	20.8	31.0	5.7
1942	26.6	29.5	4.8
1943	33.2	27.8	3.6
1944	40.3	26.1	2.6
1945	45.9	24.7	1.9
1946	44.9	27.1	1.5
1947	38.7	31.2	1.7
1948	30.2	36.6	1.9
1949	25.6	39.0	2.1
1950	21.0	39.7	2.2
1951	16.1	41.4	2.4
1952	14.0	43.1	2.6

* Source: *Life Insurance Fact Book,* 1953.

How did the complexion of the assets of life insurance companies in the United States change between the years 1945 and 1952?

6. Depict the following data graphically on one chart.

COST OF MAGAZINE ADVERTISING FOR TWO TYPES OF PRODUCTS, 1952*
(Millions of Dollars)

Month	*Beer, Wine, Liquors*	*Smoking Materials*	*Month*	*Beer, Wine, Liquors*	*Smoking Materials*
Jan.	1.6	1.1	July	2.0	1.2
Feb.	2.3	1.2	Aug.	1.7	1.4
Mar.	2.9	1.4	Sept.	2.4	1.4
Apr.	2.5	1.3	Oct.	2.9	1.5
May	2.6	1.4	Nov.	3.2	1.5
June	2.3	1.6	Dec.	4.3	1.7

* Source: *Business Statistics,* U.S. Department of Commerce, 1953.

How did cost of advertising for these two types of products differ over the year?

7. On one chart, draw line graphs for the following data.

AVERAGE WEEKLY EARNINGS IN TWO INDUSTRIES*

Year	*Manufacturing*	*Retail Trade*†	*Year*	*Manufacturing*	*Retail Trade*†
1941	$29.58	$24.42	1947	$49.97	$40.66
1942	36.65	25.73	1948	54.14	43.85
1943	43.14	27.36	1949	54.92	45.93
1944	46.08	29.53	1950	59.33	47.63
1945	44.39	31.55	1951	64.71	50.65
1946	43.82	36.35	1952	67.97	52.67

* Source: *Business Statistics*, U.S. Department of Commerce, 1953.
† Except eating and drinking places.

Discuss reasons for the rise in earnings in 1947. In what period does it appear that retail employees had the lowest wages relative to those of manufacturing employees?

8. Depict graphically the increase in the number of persons receiving monthly benefits at year's end from Federal old age and survivors insurance.

OLD AGE AND SURVIVORS INSURANCE,
NUMBER RECEIVING BENEFITS AT YEAR'S END*
(In Thousands)

Year	*Number*	*Year*	*Number*	*Year*	*Number*
1941	434	1945	1,288	1949	2,743
1942	598	1946	1,642	1950	3,477
1943	748	1947	1,978	1951	4,379
1944	955	1948	2,315	1952	5,026

* Source: *Life Insurance Fact Book*, 1953.

9. The following table gives the number of persons attending school in the United States in 1950 for the ages 6 years through 20 years.

NUMBER OF PERSONS ATTENDING SCHOOL IN THE UNITED STATES IN 1950*
(In Millions)

Age	*Number*	*Age*	*Number*	*Age*	*Number*
6	1.80	11	2.15	16	1.72
7	2.72	12	2.30	17	1.44
8	2.41	13	2.22	18	0.88
9	2.23	14	2.01	19	0.58
10	2.21	15	1.98	20	0.40

* Source: *Pocket Almanac,* 1954.

Give a possible explanation for the drop in number attending school at age 16 years and again at age 18 years. How would you account for the comparatively small number attending at age 6 years?

10. The following table gives the proportion of trade-union women members for 213 unions in the United States in 1953. Draw a smooth curve for the data.

DISTRIBUTION OF PER CENT WOMEN IN TRADE UNIONS*

Per Cent Women	*Number of Unions*
Under 10%	125
10% and under 20%	25
20% and under 30%	16
30% and under 40%	10
40% and under 50%	9
50% and under 60%	13
60% and under 70%	10
70% and under 80%	3
80% and under 90%	1
90% and over	1
	213

* Source: *Pocket Almanac,* 1954.

35. Frequency Distribution. The data in Table 1 may be put into a very compact form by arranging the number of errors for each form of the test into *classes* of equal width and then recording the number of errors occurring in each class.

The *class interval* is indicated by writing its *lower limit* and its *upper limit.* Each limit is halfway between the largest item in the lower class and the smallest item in the next higher class. In Table 1, we see that the number of errors on Form A ranges from 7 to 23. We may divide the total range into classes, say, 6 but less than 8, 8 but less than 10, and so forth. A convenient way to write these class intervals is: 5.5–7.5; 7.5–9.5; and so forth. Then by counting the items that fall within each class interval, we obtain the *frequency table* (Table 2). In this table the data are arranged according to nine intervals.

Other frequency tables for the same data can be made by making other choices of class intervals. There is no set rule for selecting the classes, but it is highly important that they be so chosen as to be fairly represented by a single value. As a whole, the nature of the data and the purpose of the investigation must determine the number of classes to be used. At this point, it is sufficient to say that the use of a few classes presents a simpler broad view of the distribution at the expense of accuracy and that the use of many classes gives greater accuracy but prevents a comprehensive view of the whole distribution.

TABLE 2

FREQUENCY DISTRIBUTION OF ERRORS MADE BY TWENTY STUDENTS ON A TYPEWRITING SPEED TEST (FORM A)

Class Limits L_1–L_2 (1)	*Class Mark* X' (2)	*Tally* (3)	*Frequency* f (4)	fX' (5)
5.5–7.5	6.5	1	1	6.5
7.5–9.5	8.5	1	1	8.5
9.5–11.5	10.5	11	2	21.0
11.5–13.5	12.5	1	1	12.5
13.5–15.5	14.5	1	1	14.5
15.5–17.5	16.5	~~1111~~	5	82.5
17.5–19.5	18.5	111	3	55.5
19.5–21.5	20.5	111	3	61.5
21.5–23.5	22.5	111	3	67.5

The *class mark* for each interval is usually taken as the average of the class limits. Thus, in Table 2 the first class mark is the average of 5.5 and 7.5, or 6.5. In general, if X' stands for the class mark, L_1 for one lower limit, and L_2 for the next lower limit,

$$X' = \frac{L_1 + L_2}{2}.$$

Table 2 is an example of a *frequency distribution*. In this particular example, the *variable* is number of errors and can have only integral values. Such a variable is usually called *discrete*. Variables such as weight, height, and so forth that can be measured to any degree of fineness are usually called *continuous*. In tabulating the data for either type of variable, it is assumed that the measurements are evenly distributed over the intervals. This assumption ordinarily leads only to a very slight error and makes possible simplification of computations.

In our example, where the data are discrete, we could have indicated the class intervals as 6–7, 8–9, and so forth, obtaining the same class marks found for the intervals selected in the table. But if data had been heights, say, measured to the nearest foot, our method of writing the intervals so that adjacent classes "join up" would take care of measurements that are not integers. In any case, the method described, and indicated in Table 2, is widely used and is suitable for discrete as well as for continuous data. Many other methods for defining class intervals are found in the literature on the subject. For this reason, it is important that there be a clear understanding as to which class an item belongs to when a frequency table is made.

PROBLEMS

1. Tabulate the number of errors made on Form B of the typewriting speed test in Table 1 into a frequency table by using class intervals of 7.5–9.5, 9.5–11.5, and so forth.

2. Make a frequency table of the Form A test errors of Table 1 by using class intervals of 6.5–9.5, 9.5–12.5, and so forth.

3. In making a spot check of berry prices in 50 city stores a reporter found the following prices per quart:

63¢	54¢	61¢	58¢	55¢
51	59	56	59	60
67	57	60	53	59
64	58	60	61	56
63	59	58	60	60
62	54	52	56	59
67	60	64	59	59
58	56	59	57	58
57	58	61	59	53
58	60	59	60	62

(*a*) Use 2-cent class intervals of 49.5–51.5, 51.5–53.5, and so forth, and make a frequency table of the berry prices.

(*b*) Use 5-cent intervals of 49.5–54.5, 54.5–59.5, and so forth. Criticize this selection of interval.

(*c*) Without grouping the data into class intervals, arrange the prices in a frequency table. Has this procedure merit in this particular problem? Explain.

4. What information of a characterizing nature can be obtained from a frequency table which cannot be obtained from the "raw" data?

5. Make a frequency distribution of the following 56 highest batting averages in a baseball league:

.316	.300	.261	.278	.268	.335	.290
.302	.280	.294	.319	.292	.283	.266
.286	.291	.288	.263	.280	.301	.290
.343	.275	.278	.317	.400	.262	.323
.277	.318	.307	.262	.305	.276	.267
.313	.309	.268	.342	.269	.266	.271
.282	.299	.281	.286	.300	.353	.288
.315	.280	.264	.275	.292	.279	.317

6. The following 50 orders for a certain drug were received by a salesman during a given week. The amount of each order is given to the nearest dollar. Make a frequency table of these order amounts, and state some uses that might possibly be made of the table.

$22	$13	$45	$38	$29	$56	$44	$72	$51	$30
10	46	32	58	43	65	49	34	66	42
31	53	16	54	34	77	54	27	44	68
40	38	74	22	19	55	27	58	28	52
39	46	63	37	48	25	47	68	36	42

7. The following 60 sales were made in the stationery department of the Real Variety department store from opening to noon on a Monday. The amounts are given to the nearest dollar. Make a frequency table of these amounts, and state at least one characteristic of the data.

$1	$4	$14	$2	$2	$5	$3	$1	$6	$4
3	1	6	3	4	3	7	3	4	3
5	3	1	4	11	10	3	11	10	3
4	7	9	3	3	1	12	4	4	6
6	3	4	7	5	4	1	8	3	2
3	8	3	4	6	3	4	3	2	4

8. Obtain from 30 or 40 individuals their weights, heights, shoe sizes, or grades on some test. Arrange the data in a frequency

table, and report the information of a characterizing nature that you have found about the individuals who cooperated in your study.

36. Histogram and Frequency Polygon. A vertical-bar graph, in which the bars are placed together, forming an outline that indicates the upper extremities of the bars, serves quite well to give a vivid picture of a frequency distribution. Such a chart is called a *histogram.* For example, the scores made by 120 applicants who took an employment aptitude test were as follows:

Score	*Number of Applicants*
29.5–39.5	8
39.5–49.5	14
49.5–59.5	26
59.5–69.5	40
69.5–79.5	22
79.5–89.5	6
89.5–99.5	4
	120

The class limits of the scores in the given frequency table are represented on the horizontal scale, and frequencies are shown on the vertical scale, the base of which is zero. Each class is then represented by a rectangle with area equal to (orproportional to) the class frequency. The histogram for the test scores is shown in Figure 2.

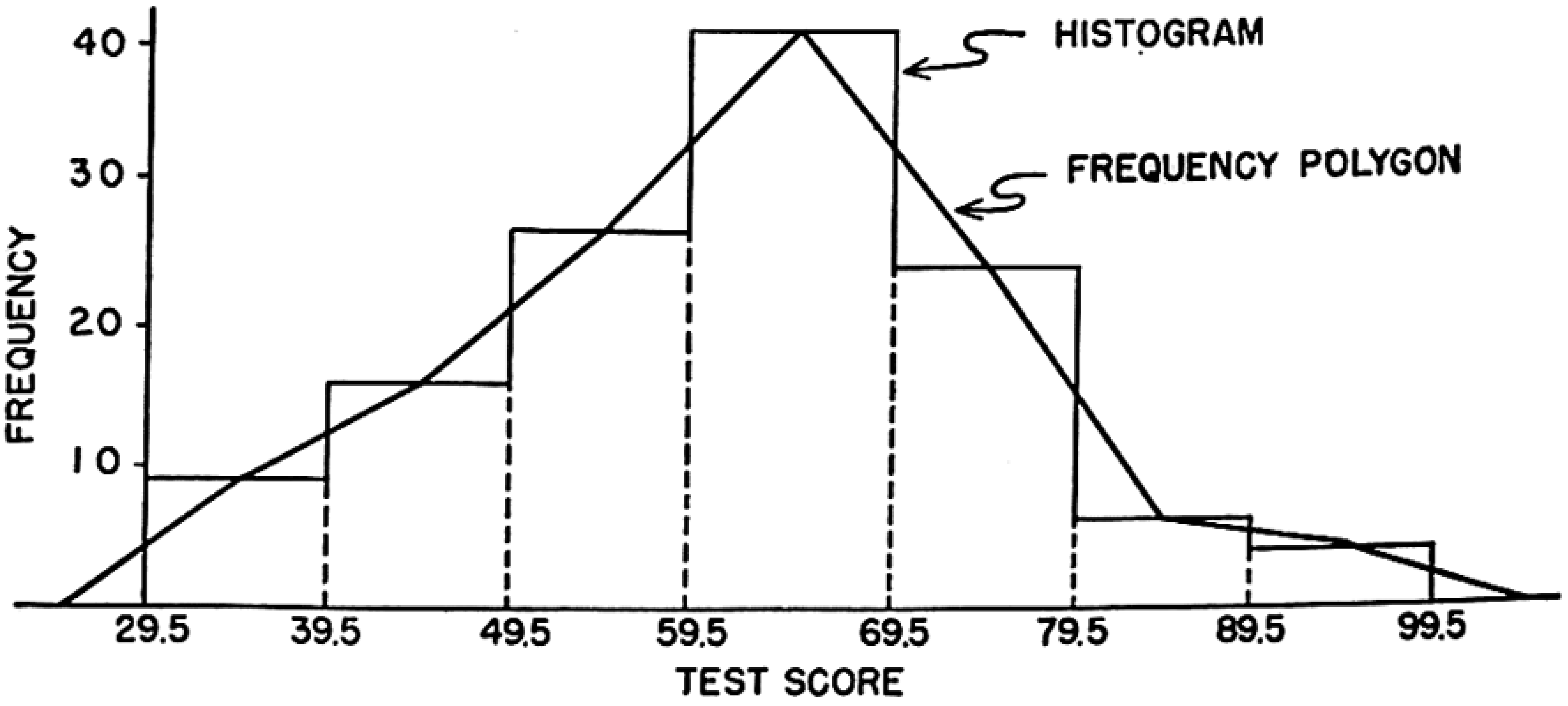


Figure 2. Frequency Distribution of Aptitude Test Scores

When the area under the histogram is taken as 1, each rectangle represents the proportion of the total frequencies falling within each class. (Should the data be given with an "open interval," as, for example, "89.5 and over," it is customary to place an arbitrary boundary on the interval so that a histogram can be constructed.)

Another way to represent a frequency distribution graphically is by means of a *frequency polygon*. This is constructed by connecting the mid-points of the tops of the rectangles of the histogram by a broken line and joining the last points at either end to the base line at the center of the next class interval. In other words, the frequency polygon passes through the upper extremities of vertical line segments representing frequencies erected at the class marks. The frequency polygon for the aptitude test scores is shown in Figure 2.

PROBLEMS

1. Construct a histogram and a frequency polygon for the distribution of errors in Table 2.

2. Construct a histogram and a frequency polygon for the number of errors made on Form B of the typewriting speed test (see Table 1, Section 34, and Problem 1, Section 35).

3. Refer to Problem 3(*a*) of Section 35, and construct a histogram and a frequency polygon for the distribution of berry prices.

4. The 1954 *Pocket Almanac* gives the following distribution of ages for the population of the United States in 1950. Construct a histogram and a frequency polygon.

AGE DISTRIBUTION OF UNITED STATES POPULATION (1950)*

Age	*Population, millions*	*Age*	*Population, millions*
0–4.99	16.3	40–44.99	10.1
5–9.99	13.2	45–49.99	9.0
10–14.99	11.4	50–54.99	8.3
15–19.99	10.7	55–59.99	7.2
20–24.99	11.3	60–64.99	6.0
25–28.99	12.1	65–69.99	5.1
30–34.99	11.6	70–74.99	3.4
35–39.99	11.2	75 and over	3.8

* SOURCE: *Pocket Almanac*, 1954.

5. Use the following data, and construct a histogram and a frequency polygon to portray the prewar size of farms in the United

States. (The intervals 3–9, 10–19, and so forth, are to be interpreted as 3–9.99, 10–19.99, and so forth.)

Number of Acres	*Number of Farms*	*Number of Acres*	*Number of Farms*
Under 3	35,573	100–174	1,404,297
3–9	535,258	175–259	540,267
10–19	683,452	260–499	473,239
20–49	1,440,143	500–999	167,452
50–99	1,444,007	1,000 and over	88,662

6. Refer to Problem 6 of Section 35, and construct a histogram and a frequency polygon for the distribution of orders.

7. Refer to Problem 7 of Section 35, and construct a histogram and a frequency polygon for the sales in the stationery department.

37. Central Tendency. When data have been gathered and arranged in tables and graphs have been made, certain questions naturally arise which cannot be answered directly from the tables, nor from the graphs. For instance, in looking at Table 1 we might ask, "What is the average number of errors made on Form A of the test? On Form B? Which item occupies the central position in each of the columns?"

The answers to questions such as these give valuable information about our data and furnish descriptive material for use in making an analysis. These questions refer to the *central tendency* of the data in that they call for measures that are to some extent representative of all the data. To answer these questions statistically, we calculate the *arithmetic mean*, the *median*, and the *mode*.

38. The Arithmetic Mean. The arithmetic mean (commonly called the *average*) is calculated as the sum of all the items divided by the number of items. Thus, the average number of errors made on Form A (Table 1) by the 20 students equals the sum of 7 + 9 + 10 + 11 + 13 + and so forth, divided by 20. The sum is found to be 333. Hence, the mean or average number of errors is 333 ÷ 20 = 16.6 (between 16 and 17 errors).

In symbols, if m stands for the arithmetic mean, X the items to be averaged, and n the number of these items, the formula for the arithmetic mean is

$$m = \frac{\Sigma X}{n}, \tag{1}$$

where the Greek letter Σ, known as the summation symbol, means that all items are to be summed.

If the data to be averaged are arranged in a frequency table, the formula for computing the arithmetic mean is essentially the same. We must remember, however, that in this case each item is assumed to be represented by its class mark (X'). Furthermore, each class mark must be multiplied by the appropriate frequency f so as to account for all the n items in making the summation. The formula, therefore, may be written

$$m = \frac{\Sigma f X'}{n}. \tag{2}$$

To continue with the example, we now refer to Table 2. The formula requires that we sum the last column in the table, 6.5 + 8.5 + 21.0 + 12.5, and so forth, and divide this sum by 20. We have

$$m = \frac{\Sigma f X'}{n} = \frac{330}{20} = 16.5.$$

The value of the arithmetic mean found by using formula (2) is an approximation and will usually differ from the exact value computed from the original values of the items. In our example, the exact average (to one decimal) was found to be 16.6. It should be pointed out that the two methods tend to yield more nearly the same results as the number of items increases and as the class interval decreases. For most purposes the computation of the arithmetic mean from grouped data will be found to be sufficiently accurate.

PROBLEMS

1. Compute the average number of errors made by the 20 students on Form B of the typewriting speed test, by averaging the figures in column (3) of Table 1. Then find the mean by arranging the data in a frequency table using intervals of 7.5–10.5, 10.5–13.5, and so forth. Compare and explain the results.

2. Find the average price of berries in 50 city stores from the data of Problem 3(*b*), Section 35.

3. The following table summarizes the weekly sales and the number of years of experience for a group of 12 salesmen in the Central Market. Find the arithmetic mean for each column of figures.

	X_1	X_2
Employees	*Weekly Sales, thousands*	*Number Years of Experience*
A	$ 4	3
B	3	1
C	7	5
D	8	5
E	12	6
F	10	8
G	3	3
H	5	4
I	7	4
J	10	11
K	13	8
L	7	5

4. The following table shows the distribution of hourly wages of 500 employees in a certain factory. Find the average hourly wage.

*Hourly Wage**	*Number of Employees*	*Hourly Wage*	*Number of Employees*
$1.40–$1.45	12	$1.65–$1.70	107
1.45– 1.50	32	1.70– 1.75	71
1.50– 1.55	40	1.75– 1.80	89
1.55– 1.60	55	1.80– 1.85	17
1.60– 1.65	60	1.85– 1.90	17

* The hourly wage intervals are to be interpreted $1.395–$1.445, $1.445–$1.495, and so forth.

5. A caterer purchased the following articles in the quantity and at the prices indicated. Find the average price per pound for each article.

Butter		*Coffee*		*Meat*		*Beans*	
Pounds	*Price*	*Pounds*	*Price*	*Pounds*	*Price*	*Pounds*	*Price*
400	$0.78	824	$1.14	56	$0.80	100	$0.34
120	0.82	135	1.30	135	0.78	230	0.37
62	0.85	1243	1.12	259	0.79	64	0.32
275	0.81	589	1.08	36	0.74	25	0.38
43	0.77	164	0.96	20	0.77	49	0.35

6. The following are the number of castings polished in 1 hr by 40 employees of the Perry Iron Company. Make a frequency table, and compute the average number of castings polished in one hour.

32 35 38 40 31 34 31 32 36 36
33 34 35 36 35 40 36 32 37 39
41 43 36 42 35 35 33 33 36 37
37 36 37 36 35 36 37 33 36 35

7. The following grades were made by 100 college students on a placement examination in mathematics. Make a frequency table, and compute the arithmetic mean. (Use class intervals of 37.5–43.5, 43.5–49.5, and so forth.)

72 72 72 96 92 92 88 84 84 84 82 80 78 78 76
74 74 74 74 72 48 64 38 47 71 63 71 62 63 54
67 69 49 60 49 69 85 66 71 68 93 77 70 58 66
68 67 79 69 55 93 45 62 51 87 70 39 71 63 83
68 73 69 67 67 85 97 63 75 50 79 68 61 66 71
65 69 75 48 81 72 75 62 54 70 73 44 70 55 85
71 70 61 73 62 47 46 59 70 60

8. The following frequency table shows the sales records of 400 clerks in a department store for a given day. Find the average sales for the day. (Sales were compiled to the nearest dollar. Interpret the intervals as 49.5–79.5, 79.5–109.5, and so forth.)

Sales, Dollars	*Number of Clerks*	*Sales, Dollars*	*Number of Clerks*
50–80	51	290–320	7
80–110	110	320–350	8
110–140	83	350–380	6
140–170	64	380–410	5
170–200	14	410–440	6
200–230	23	440–470	5
230–260	9	470–500	3
260–290	6		

9. Count the number of letters in each word of the first two paragraphs of this chapter, and make a tally sheet showing the number of one-letter words, two-letter words, and so forth. Draw a

frequency polygon, and compute the mean number of letters per word.

10. List the two right-hand digits in a column of a telephone directory until 100 such two-digit numbers have been obtained. Group these two-digit numbers into classes 00–09, 10–19, and so forth. Make a histogram for the data, and compute the mean. In what respect does this histogram appear to differ from most of the other histograms you have made in this chapter?

39. Short-cut Method of Calculating the Arithmetic Mean. As we might expect, the arithmetic mean is equidistant from the combined items above it and the combined items below it. Thus, the deviations of the items 8, 5, and 14, from their average 9, are 1 and 4 below the mean and 5 above the mean. That is, if m stands for the mean; X, an item; and d, a deviation; and if we consider deviations above the mean to be positive and those below the mean negative, we have:

$$\begin{aligned} X - m &= \quad d \\ 8 - 9 &= -1 \\ 5 - 9 &= -4 \\ 14 - 9 &= \underline{+5} \\ \text{Total} &= \quad 0 \end{aligned}$$

Expressed algebraically, $\Sigma d = 0$. In words, the sum of the deviations from the mean equals zero. This property of the arithmetic mean which follows directly from the definition of the mean makes possible a method of computing the mean which is often convenient to use where a large number of items is involved.

Suppose, in the example above, we had arbitrarily taken 5 to be a rough approximation to the mean of the three items. Then the deviations from this arbitrary starting point are seen to be 3, 0, and 9. The average of these deviations is $(3 + 0 + 9) \div 3 = +4$. This tells us that the true mean of the items is actually $+4$ greater than the 5 which we had taken as a rough approximation and that the $+4$ is a correction which must be added to the 5 to give 9, the correct mean. This process of obtaining the mean may be expressed by the formula

$$m = m_a + c,$$

where m_a is the assumed mean arbitrarily selected and c is the correction which is the algebraic sum of the deviations about the assumed mean divided by the number of items. Since $c = (\Sigma d)/n$, or $(\Sigma fd)/n$ in the case of grouped data, the formula may be written either

$$m = m_a + \frac{\Sigma d}{n}, \qquad \text{or} \qquad m = m_a + \frac{\Sigma fd}{n}. \tag{3}$$

This short-cut method may also be used when the data are arranged in a frequency table. We may simply select one of the class marks as the assumed mean and express the deviations of the other class marks from it. Then these interval deviations are multiplied by the corresponding frequencies and averaged. Upon multiplying by the width of the class interval, we obtain the correction which must be added algebraically to the assumed mean. We shall illustrate the process by using the data of Table 2.

Class Limits L_1–L_2	*Class Mark* X'	*Frequency* f	*Interval Deviation* d	fd
5.5–7.5	6.5	1	−6	−6
7.5–9.5	8.5	1	−5	−5
9.5–11.5	10.5	2	−4	−8
11.5–13.5	12.5	1	−3	−3
13.5–15.5	14.5	1	−2	−2
15.5–17.5	16.5	5	−1	−5
17.5–19.5	18.5	3	0	0
19.5–21.5	20.5	3	1	3
21.5–23.5	22.5	3	2	6
		20		20)−20
				−1

Assumed mean $= m_a = 18.5$; $c = \dfrac{\Sigma fd}{n} = \dfrac{-20}{20} = -1$

Multiply by class interval, $\times\ 2$

Correction $= -2$

Mean $= m_a + c = 18.5 - 2 = 16.5$.

It will be observed from the foregoing illustrations that if the assumed mean is taken too small, or less than the actual mean, the correction will turn out to be positive, and if taken larger than the actual mean, the correction will turn out to be negative. In either case, the formula $m = m_a + c$ holds when it is understood that the correction is to be added algebraically—that is, with due regard to sign.

PROBLEMS

1. The following frequency table summarizes the number of savings bonds purchased by 80 employees in a certain retail store. Compute the average number of bonds purchased (*a*) by formula (2) and (*b*) by formula (3).

Number of Bonds (Interval)	*Employees Buying Bonds (f)*
−0.5–4.5	35
4.5–9.5	20
9.5–14.5	12
14.5–19.5	8
19.5–24.5	3
24.5–29.5	2

2. Use the short-cut method, and calculate the arithmetic mean for the number of errors made on Form B of the typewriting speed test (Table 1). (Take class intervals of 7.5–9.5, 9.5–11.5, and so forth.)

3. Find in two ways the mean of the following price relatives:

142	176	186	134	105	156	249	197	294	181
140	99	156	138	256	191	136	186	188	167
205	180	73	179	210	243	235	256	179	168
144	117	116	227	227	114	175	175	113	169
149	105	133	295	215	84	273	137	161	189
183	114	195	216	198	209	185	177	156	140

4. Use the short-cut method, and compute the mean hourly wage for the data of Problem 4, Section 38.

5. Use the short-cut method, and compute the average daily sales for the department-store data of Problem 8, Section 38.

6. Use the short-cut method, and compute the mean per cent of nickel for the following distribution of ore samples:

Per Cent Nickel (X)	*Number of Samples* (f)
1.5–4.5	20
4.5–7.5	18
7.5–10.5	14
10.5–13.5	9
13.5–16.5	6
16.5–19.5	3

40. The Median. Additional information about business data can be obtained from a knowledge of that item which occupies the central position when the items have been arranged in order of magnitude. For example, five loan companies were found to be charging annual rates of 6, 6.75, 7.25, 8, and 28.25 per cent, respectively. What per cent is representative of the rates?

We might, of course, compute the average of the five rates, which in this case turns out to be 11.25 per cent. But we see that this average has been greatly affected by the relatively large rate of 28.25 per cent and thus would perhaps not be typical of all the rates. For this reason, we might wish to select the *middle* item, 7.25 per cent, as that rate most typical for all five companies. In such cases as this, where the items are arranged in order of size, the central item is called the *median.* If there is an even number of items in the list, the median is taken to be the average of the two middle items. Thus, the median of the prices 12, 15, 16, 17, 19, 20, 24, and 30 cents is the average of 17 and 19 cents, or 18 cents.

Illustration: Find the median number of errors made by 20 students on Form A of the typewriting speed test (Table 1).

First, we arrange the data in order of magnitude:

7, 9, 10, 11, 13, 15, 16, 16, 16, 17, 17, 18, 19, 19, 20, 20, 21, 23, 23, 23.

Since there are 20 items, the median is taken to be the average of the two middle values. Hence, the median is $(17 + 17) \div 2$, or 17.

Problems

1. Find the median number of errors for Form B of the test (Table 1).

2. Find the median of the following numbers: 13, 18, 22, 14, 19, 13, 16, 21, 15, 18, 20, 19, 15, 16, 23.

3. Find the median of the following scores: 89, 92, 77, 80, 83, 78, 91, 88, 76, 89, 75, 81, 89, 79.

4. Find the median for the baseball data of Problem 5, Section 35.

5. Find the median of the following 24 golf scores: 84, 78, 86, 80, 72, 92, 83, 76, 78, 71, 77, 89, 104, 77, 72, 80, 84, 72, 89, 92, 78, 83, 104, 84.

6. Find the median age for 27 traffic officers whose ages are as follows: 26, 25, 31, 33, 28, 30, 29, 25, 27, 36, 57, 29, 35, 50, 35, 31, 25, 27, 29, 31, 30, 25, 29, 30, 25, 64, 30.

7. The annual salaries paid 20 professors in a certain college were $4,200, $6,000, $5,400, $6,400, $4,600, $5,200, $6,000, $4,400, $8,000, $5,600, $5,200, $6,000, $4,600, $5,200, $5,800, $5,600, $5,000, $5,200, $5,600, $5,400. Find the median salary.

8. The ages of 350 candidates taking a state bar examination were as follows. Find the median age of the candidates.

Age	*Number*	*Age*	*Number*
20	1	26	20
21	4	27	28
22	18	28	17
23	102	29	4
24	97	32	5
25	52	36	2

41. The Median from Grouped Data. When the data have been arranged in a frequency table showing the class intervals, we first determine the class in which the median falls. Then, we make the assumption that the values within an interval are uniformly distributed and find by simple interpolation the position within the interval occupied by the central item. For example, let us refer to the distribution of typing errors (Table 2). According to the meaning of median, there must be $\frac{20}{2}$ or 10 items below and 10 items above the median. If, therefore, we start at the first interval and add up the frequencies, we find that the tenth item falls in the interval 15.5–17.5. We know, then, that the median is somewhere between 15.5 and 17.5. But what is its exact position?

If we count the items up to, but not including, those in the median interval, we find 6 items. We must count 4 more items before reaching the median. Since there are 5 items in the median interval, we must move $\frac{4}{5}$ of the distance through this interval before reaching the median position. The interval is 2 units in length. Hence, $\frac{4}{5}$ of 2, or 1.6, added to the 15.5 gives 17.1 as the median. The reader will observe that the median calculated from the grouped data is 17.1, whereas it was found to be 17 when computed from the ungrouped data. This difference is due, of course, to the assumption of uniform distribution within intervals.

Illustration: Find the median of the following scores made by 80 students who took the Self-Test at the end of Chapter II.

Interval	*Frequency*
79.5–89.5	8
69.5–79.5	16
59.5–69.5	27
49.5–59.5	16
39.5–49.5	8
29.5–39.5	5
	80

Counting from the bottom, we find 29 items up to the 59.5–69.5 interval. There remain 11 items to account for 40, or half, of the total. The median interval contains 27 items. Hence, the median equals 59.5 plus $\frac{11}{27}$ of 10, or $59.5 + 4.1 = 63.6$

PROBLEMS

1. Find the median number of bonds purchased for the data of Problem 1, Section 39.

2. Find the median monthly hours worked overtime by 2,460 employees of the General Cork Company distributed as follows:

Monthly Hours Overtime	*Number of Employees*	*Monthly Hours Overtime*	*Number of Employees*
Under 5 hr*	27	20 and under 25 hr	962
5 and under 10 hr	209	25 and under 30 hr	83
10 and under 15 hr	522	30 and under 35 hr	10
15 and under 20 hr	641	35 and under 40 hr	6
			2,460

---

* Assume that overtime is given to the nearest hour, and interpret intervals as −0.5–4.5, 4.5–9.5, and so forth.

3. The following distribution shows the test scores made by 130 applicants taking a trade skill test. Find the median score.

Trade Test Score	*Number of Applicants*	*Trade Test Score*	*Number of Applicants*
89.5–99.5	3	39.5–49.5	26
79.5–89.5	7	29.5–39.5	9
69.5–79.5	20	19.5–29.5	3
59.5–69.5	21	9.5–19.5	2
49.5–59.5	38	−0.5–9.5	1

4. The following numbers represent the errors made by 96 girls in sorting cartridges over a period of a week. Group the errors in intervals of four units beginning with the smallest, and compute both the median and the mean from the resulting frequency distribution. Which measure is more appropriate for describing the general nature of the whole distribution? Why?

5	8	13	17	26	21	28	16	11	5	10	4	10	12	15	4
24	7	8	8	9	11	10	11	6	11	9	8	11	9	27	10
5	9	6	9	7	7	18	7	9	6	5	10	4	10	9	11
9	4	4	6	4	11	8	10	5	8	17	9	6	8	4	8
10	9	5	11	10	6	9	7	11	10	9	11	9	11	26	7
4	8	8	5	8	9	7	9	6	7	10	6	10	6	5	25

5. Given the following frequency distribution, compute the median of the continuous variable X. Mention several quantities for which X might conceivably stand.

X	f
1–2.99	3
3–4.99	8
5–6.99	21
7–8.99	14
9–10.99	8

6. Given the following frequency distribution, compute the median of the continuous variable X.

X	f
21–25.99	72
26–30.99	131
31–35.99	158
36–40.99	113
41–45.99	45
46–50.99	38

7. Find the median hourly wage for the data of Problem 4, Section 38.

8. Find the median berry price for the data of Problem 3(*b*), Section 35.

9. Find the median of the sales records of the 400 clerks in Problem 8, Section 38.

10. The following table gives the age distribution of the male population of the United States in 1950. Find both the median and the mean age. What interpretation would you give to the population figure for the age group 15–19?

Age Group, Years	*Population, thousands*	*Age Group, Years*	*Population, thousands*
0–4	8,301	55–59	3,608
5–9	6,825	60–64	3,029
10–14	5,680	65–69	2,364
15–19	5,302	70–74	1,610
20–24	5,457	75–79	1,027*
25–29	5,924	80–84	483*
30–34	5,735	85–89	169*
35–39	5,476	90–94	42*
40–44	5,029	95–99	13*
45–49	4,520	100–104	3*
50–54	4,036		

* Estimated by authors. Other figures taken from the *Pocket Almanac* for 1954.

42. The Mode. Under certain circumstances it is more valuable to name that item or measure in a distribution which occurs most frequently than it is to compute either the mean or the median. Particularly is this true when we are interested in that item which seems to typify a very large portion of any given group.

This would be the case, for example, where a sales campaign is planned with the intention of reaching the population of a given community. The income group containing the most people would probably be the best guide as to the class of goods to be advertised. That measure or item which occurs most frequently in a distribution is called the *mode,* and the group having the most items is called the *modal group.*

Illustrations:

1. A survey of the annual insurance premiums paid by the 30 employees of the Modern Make-Up Company showed the following payments:

Annual Premium, Dollars	*Number of Employees*	
4	1	
5	3	
8	18	(Mode = $8)
11	3	
18	2	
24	2	
40	1	

The $8 premium, which occurs more frequently than any other, is called the mode.

2. The following sales of a certain commodity were noted in the stocks edition of a local newspaper. What is the modal group that indicates the bulk of sales?

Price, Dollars	*Number*	
4–6	31	
6–8	127	
8–10	462	(Modal group = $8–$10)
10–12	203	
12–14	84	

It is seen that the bulk of sales is in the $8 to $10 group. There are several ways by which the modal point may be defined within the interval, but in most business situations it is sufficient to know the modal group. Methods of interpolation for the mode are usually based upon the relationship of the frequency of the modal class to the frequencies of adjacent classes. These methods are not considered important enough to be included in this discussion.

PROBLEMS

1. Find the modal group for each of Problems 2 and 3 of the preceding section.

2. Find the modal group for each of Problems 5, 6, and 10 of the preceding section.

3. Given the following scores, calculate the mean, median, and mode, and compare these measures: 80, 60, 75, 50, 65, 45, 60, 30, 55, 60, 85, 65, 70, 40, 60, 55, 35, 65.

4. Calculate the mean, median, and mode for the following items, and compare them: 10, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

43. Dispersion. We have seen that the mean, the median, and the mode are representative quantities that may be used to describe the general nature of a distribution as a whole. That is, they are measures of central tendency indicating something of the manner in which the items tend to cluster together and are therefore valuable descriptive quantities. But now let us consider the following two sets of measurements:

(1) 25, 26, 27, 28, 29, 30, 30, 30, 31;
(2) 1, 2, 5, 7, 15, 29, 30, 30, 30, 37, 126.

It is seen that the median of each set is 29 and that the mode of each set is 30. The mean of the first set is found to be (25 + 26 + and so forth) ÷ 9 = 28.4. Also, the mean of the second set is found to be (1 + 2 + 5 + and so forth) ÷ 11 = 28.4. The two sets of data, therefore, have the same characteristics as far as measures of central tendency are concerned. But it is apparent that the items of the second set, which range from 1 to 126, are more widely scattered, or dispersed, from the average than are those of the first set, which range from 25 to 31. It would be useful, therefore, to have some measure of the extent to which the items in each set depart from the central tendency, that is, some measure of *dispersion*, so that we may have further means of describing the data and also of comparing the two sets of data.

44. Average Deviation. For each of the two sets of numbers (1) and (2) in Section 43, the mean was found to be 28.4. In each, some of the numbers are above the mean, and others are below it. Those above the mean are said to deviate positively from the mean, whereas those below the mean are said to deviate negatively. Thus, in (1) the number 30 deviates 30 − 28.4, or +1.6, from the mean, and the number 26 in the same set deviates 26 − 28.4, or −2.4, from the mean. In general, if X stands for any item in the data, m the mean, and d a deviation, then

$$d = X - m.$$

The numbers and their corresponding deviations for (1) and (2) may now be written:

(1)		(2)	
Item X	*Deviation* d	*Item* X	*Deviation* d
25	−3.4	1	−27.4
26	−2.4	2	−26.4
27	−1.4	5	−23.4
28	−0.4	7	−21.4
29	+0.6	15	−13.4
30	+1.6	29	+ 0.6
30	+1.6	30	+ 1.6
30	+1.6	30	+ 1.6
31	+2.6	30	+ 1.6
Sum: 256	+0.4	37	+ 8.6
		126	+97.6
		Sum: 312	−0.4
Mean = 256 ÷ 9 = 28.4		Mean = 312 ÷ 11 = 28.4	

We see that for both sets of numbers the sum of the deviations from the mean is not exactly zero. This is, of course, as it should be because of the way in which each mean was calculated. Had we used an exact mean instead of one correct to tenths, the sum of the deviations would necessarily have been zero. But if we confine our interest to the magnitude of the deviations without regard to sign (absolute values), we may average them and thus obtain a measure of the dispersion of the numbers from the mean of all the numbers. The average of the absolute values of the deviations from the mean is called *average deviation* (A.D.). Thus, for each of the above sets we have:

$$\text{For (1) Average Deviation} = \frac{3.4 + 2.4 + \cdots + 1.6 + 2.6}{9} = 1.7.$$

$$\text{For (2) Average Deviation} = \frac{27.4 + 26.4 + \cdots + 8.6 + 97.6}{11}$$

$$= 20.3.$$

The average deviation for the first set of numbers is 1.7, whereas for the second set of numbers the average deviation is 20.3. This tells us that there is a much greater scatter, or dispersion, for the second set than for the first.

To indicate that the absolute values of numbers are to be taken, vertical bars are usually placed about the numbers. Thus, if d is a deviation from the mean, $|d|$ stands for the absolute value of the deviation. The formula for average deviation may accordingly be written

$$\text{A.D.} = \frac{\Sigma|d|}{n}, \tag{4}$$

in which the Σ indicates that all the deviations are to be summed and n is the number of items.

If the data are grouped in a frequency table, the d's are taken as deviations of the class marks from the mean. These deviations are then multiplied by their respective frequencies and averaged. The formula in this situation becomes

$$\text{A.D.} = \frac{\Sigma f|d|}{n}. \tag{5}$$

Illustration: Find the average deviation for the following distribution of ages:

Age	X'	f	fX'	d	fd
9.5–11.5	10.5	3	31.5	−3.7	−11.1
11.5–13.5	12.5	15	187.5	−1.7	−25.5
13.5–15.5	14.5	20	290.0	0.3	6.0
15.5–17.5	16.5	10	165.0	2.3	23.0
17.5–19.5	18.5	2	37.0	4.3	8.6
		50	50)711.0		−36.6
					+37.6
					50) 74.2
		Mean = 14.2			A.D. = 1.48

Other short-cut formulas for finding average deviation in terms of deviations from an assumed mean and in terms of the "raw" data are given in standard statistical texts. These methods are very convenient in working with a large number of items or with numbers involving decimals and are especially useful for computation with the aid of a calculating machine.

PROBLEMS

1. Find the average deviation for the following 12 stock quotations: 1.40, 1.65, 1.70, 1.50, 1.48, 1.72, 1.55, 1.37, 1.69, 1.50, 1.53, 1.50.

2. Compute the average deviation for the number of errors made on Form A of the typewriting speed test (Table 1).

3. In target practice two marksmen turned in the following scores:

Marksman A..........	92	95	90	98	91	87	95	96	88	93
Marksman B..........	93	98	86	93	92	98	94	94	93	91

Which marksman was the more consistent? Why?

4. Calculate the average deviation for the scores: 27, 31, 29, 33, 34, 30.

5. Calculate the average deviation for the scores: 3, 8, 7, 7, 4, 6, 5, 5.

6. At a certain gathering it was decided to record the tardiness in number of minutes for each individual as he arrived, with the following results for 53 persons in attendance:

Number of Minutes Tardy	*Frequency*
20–24.9	3
15–19.9	8
10–14.9	20
5–9.9	12
0–4.9	10
	53

What was the average number of minutes that the individuals were tardy? What was the average deviation? What use could the chairman make of a knowledge of these measures in reprimanding the group?

7. Find the average deviation for the data of Problem 4, Section 42.

8. Find the average deviation for the data of Problem 3, Section 42.

9. Find the average deviation for the data of Problem 3, Section 40.

10. Find the average deviation for the data of Problem 2, Section 40.

11. The monthly production of sparkling wines in the United States in thousands of wine gallons is given in the following table for the year 1952. Find (*a*) the mean and (*b*) the average deviation of production from the mean.

Month	*Production*	*Month*	*Production*
Jan.	141	July	63
Feb.	67	Aug.	100
Mar.	55	Sept.	62
Apr.	201	Oct.	90
May	129	Nov.	82
June	102	Dec.	77

12. Find the average deviation for ages of traffic officers by using the data of Problem 6, Section 40.

45. Standard Deviation. Let us now return to the two sets of data with which we began our discussion of dispersion and for which average deviations were computed in the previous section. We have:

(1) 25, 26, 27, 28, 29, 30, 30, 30, 31;
(2) 1, 2, 5, 7, 15, 29, 30, 30, 30, 37, 126.

These two sets of data have the same characteristics as far as measures of central tendency are concerned but are quite unlike in the extent of scatter about any such measure. We have already learned that the mean for each set is 28.4 and that the average deviations are 1.7 for (1) and 20.3 for (2) (see Section 44).

Instead of averaging the absolute values of the deviations, a measure of dispersion may be obtained by first squaring the deviations, averaging these squares, and then extracting the square root of the result. The measure of dispersion obtained in this manner is called the *standard deviation* (S.D.). It is perhaps the most important measure of dispersion used in the field of statistics and is usually designated by the Greek letter σ (sigma). The calculation of the standard deviation for the sets of numbers (1) and (2) is as follows:

SET (1)

Item X	*Deviation* d	*Deviation Squared* d^2
25	−3.4	11.56
26	−2.4	5.76
27	−1.4	1.96
28	−0.4	0.16
29	+0.6	0.36
30	+1.6	2.56
30	+1.6	2.56
30	+1.6	2.56
31	+2.6	6.76
Sum of squared deviations...		34.24
Average of squared deviations....................		3.80
Square root (σ)............		1.9

SET (2)

Item X	*Deviation* d	*Deviation Squared* d^2
1	−27.4	750.76
2	−26.4	696.96
5	−23.4	547.56
7	−21.4	457.96
15	−13.4	179.56
29	+ 0.6	0.36
30	+ 1.6	2.56
30	+ 1.6	2.56
30	+ 1.6	2.56
37	+ 8.6	73.96
126	+97.6	9,525.76
Sum of squared deviations		12,240.56
Average of squared deviations.............		1,112.78
Square root (σ)........		33.4

The standard deviation for set (1) is found to be 1.9. For the items of set (2) the standard deviation is 33.4. This means that the second set of numbers is much more widely scattered, or dispersed, than are the numbers in the first set. This is exactly the same information obtained from a knowledge of the average deviation of each set. It should be pointed out that the standard deviation has certain mathematical advantages over other measures of dispersion which make it important in the field of statistics. As the foregoing illustration would lead us to conclude, the formula for standard deviation may be written

$$\sigma = \sqrt{\frac{\Sigma d^2}{n}}, \tag{A}$$

in which d is a deviation from the mean, n is the number of items, and Σ means to sum all the squared deviations.

If we substitute $X - m$ for d in formula (A) and carry out certain algebraic steps, we may obtain the following formula for standard deviation in terms of the original items (X) of the data:

$$\sigma = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2}. \tag{B}$$

Formula (B) will give precisely the same result as formula (A), provided that the exact mean has been used. In the examples just given, the mean was rounded off to the first decimal place, so that the obtained values are slightly in error. The computation of the standard deviation of the items in set (1) by the use of formula (B) is as follows:

$$\frac{\Sigma X^2}{n} = \frac{(25^2 + 26^2 + 27^2 + 28^2 + 29^2 + 30^2 + 30^2 + 30^2 + 31^2)}{9}$$

$$= \frac{7{,}316}{9} = 812.89.$$

$$\frac{\Sigma X}{n} = \frac{(25 + 26 + 27 + 28 + 29 + 30 + 30 + 30 + 31)}{9}$$

$$= \frac{256}{9} = 28.44.$$

$$\left(\frac{\Sigma X}{n}\right)^2 = (28.44)^2 = 808.83$$

$$\sigma = \sqrt{\frac{\Sigma X^2}{n} - \left(\frac{\Sigma X}{n}\right)^2} = \sqrt{812.89 - 808.83}$$

$$= \sqrt{4.06} = 2.01.$$

The standard deviation by using formula (B) is found to be 2.01, whereas it turned out to be 1.9 by using formula (A) where the mean was rounded off to the first decimal place. Although one must deal with larger numbers when calculating the standard deviation directly from the "raw scores" of the data, one will find formula (B) advantageous if a calculating machine or table of squares is available. When the items are arranged in a frequency distribution, it will be recalled (see Section 35) that they are considered to be centered about the mid-point (X') of the interval in which they occur. The formula for standard deviation in this situation becomes:

$$\sigma = \sqrt{\frac{\Sigma fX'^2}{n} - \left(\frac{\Sigma fX'}{n}\right)^2}. \qquad (C)$$

If we carry out the work in terms of intervals to avoid the use of the unnecessarily large numbers of the original data and express the class intervals as deviations from an assumed mean, the work of calculating a standard deviation becomes greatly abbreviated. If d stands for an interval deviation and i the width of the interval, our formula becomes

$$\sigma = \left[\sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2}\,\right] i. \qquad (D)$$

Illustration: Find the standard deviation for the distribution of errors made by 20 students on a typewriting speed test (see Table 2, Section 35).

Solution: The necessary calculations for the use of formula (*D*) follow:

Interval ($i = 2$)	*Deviation* d	*Frequency* f	fd	d^2	fd^2
5.5–7.5	−6	1	−6	36	36
7.5–9.5	−5	1	−5	25	25
9.5–11.5	−4	2	−8	16	32
11.5–13.5	−3	1	−3	9	9
13.5–15.5	−2	1	−2	4	4
15.5–17.5	−1	5	−5	1	5
17.5–19.5	0	3	0	0	0
19.5–21.5	1	3	3	1	3
21.5–23.5	2	3	6	4	12
Totals (Σ)		20	−20		126

$$\sigma = \left[\sqrt{\frac{\Sigma fd^2}{n} - \left(\frac{\Sigma fd}{n}\right)^2}\,\right] i = \left[\sqrt{\frac{126}{20} - \left(\frac{-20}{20}\right)^2}\,\right](2)$$

$$= \sqrt{6.3 - 1}\,(2) = \sqrt{5.3}\,(2) = 4.6.$$

PROBLEMS

1. Find the standard deviation of the distribution of typewriting speed test errors (Form A, Table 1, Section 34) without grouping the data. Why is your answer different from the one obtained by grouping the data?

2. Find the standard deviation of the following retail prices:

1.50 1.60 1.48 1.40 1.50 1.52 1.65 1.53 1.36 1.47 1.61
1.49 1.53 1.26 1.26 1.74 1.76 1.68 1.24 1.43 1.51 1.46
1.50 1.55 1.73 1.73 1.33 1.37 1.38 1.42 1.46 1.45 1.52
1.57 1.78 1.57 1.57 1.51 1.80

3. Find the standard deviation of the following hourly wages:

Hourly Wage, Cents	*Number of Employees*
149.5–159.5	60
139.5–149.5	89
129.5–139.5	132
119.5–129.5	156
109.5–119.5	105
99.5–109.5	95
89.5–99.5	73

4. Calculate the standard deviation of the following distribution of ages:

Interval	*f*
2.5–5.5	4
5.5–8.5	7
8.5–11.5	12
11.5–14.5	13
14.5–17.5	5

5. Calculate the standard deviation of the following distribution of egg prices:

Interval	*f*
69.5–73.5	3
73.5–77.5	6
77.5–81.5	15
81.5–85.5	40
85.5–89.5	16
89.5–93.5	9
93.5–97.5	8

6. Find the standard deviation for the data of Problems 5 and 6, Section 41.

7. Find the standard deviation of the monthly hours overtime of 2,460 employees by using the data of Problem 2, Section 41.

8. Find the standard deviation of the scores of the 130 applicants given in Problem 3, Section 41.

9. Find the standard deviation of the price relatives for the data of Problem 3, Section 39.

10. Find the standard deviation of the hourly wages of the 500 workers as given in Problem 4, Section 38.

11. Forty-two members of Beta Beta Kappa fraternity made the following grade point averages for the first semester of the 1954–1955 college year. Find the mean and the standard deviation of the grade point averages.

2.39	3.15	3.01	2.54	2.29	1.87
2.49	2.90	3.03	2.38	2.23	1.84
2.41	3.00	3.16	2.91	2.11	1.87
2.75	3.11	3.18	2.67	1.94	1.73
2.69	3.08	3.03	2.53	1.72	1.57
2.87	3.09	2.69	2.63	1.74	1.62
2.79	3.26	2.98	2.48	1.91	1.47

46. Point Measures of Dispersion. Both the average deviation and the standard deviation measure dispersion, or scatter, in terms of deviations from the mean. Another very easy and meaningful method is to consider the scatter in terms of distances between points in the distribution of the data. This is the case when we talk of the *range*, the *quartile deviation*, the *difference between percentile points* and so forth. Since the technique of locating any of the required points is precisely the same as that for locating a median, we shall define the most frequently used interpoint measures of dispersion and illustrate their computation.

(1) *The Range.* The distance from the lowest score to the highest is called the *range.* Thus, in the series of numbers 23, 34, 56, 72, 92, we may say that the numbers range from 23 to 92, or simply that the range is 69.

(2) *The Quartile Deviation.* In computing the median, we find that point above which lie half of the cases and below which lie the other half. By the same technique we may find that point which is $\frac{1}{4}$ of the way through the number of cases. This point is called the first quartile, or Q_1. The point which is $\frac{3}{4}$ of the way through the distribution is called the third quartile and is labeled Q_3. The distance from Q_1 to a point halfway to the point Q_3 is known as the *quartile deviation,* sometimes called the *semi-interquartile range,* and is denoted by Q. The formula is

$$Q = \frac{Q_3 - Q_1}{2}. \tag{6}$$

(3) *Percentiles.* The *percentile points* divide the distribution into 100 parts. Thus, the tenth percentile is that point below which lie $\frac{1}{10}$ of the cases and above which lie $\frac{9}{10}$ of the cases; the 98th percentile is that point below which lie $\frac{98}{100}$ of the cases and above which lie $\frac{2}{100}$ of the cases. All the percentile points may be obtained, one by one, by the same technique used in finding the median, but ordinarily a few are located and the others are approximated by interpolation.

Illustration: Scores made on a college entrance examination by 200 students were as follows:

Score	*Frequency*	*Cumulative Frequency*
89.5–100.5	2	200
79.5–89.5	15	198
69.5–79.5	33	183
59.5–69.5	48	150
49.5–59.5	54	102
39.5–49.5	33	48
29.5–39.5	12	15
19.5–29.5	3	3

Find the range, quartile deviation, and 95th percentile.

Solution: The range is, presumably, $100 - 20 = 80$. To find Q_1, count $\frac{1}{4}$ of 200, or 50, cases beginning at the bottom. It is seen from the cumulative-frequency column that the first quartile point must lie in the interval 49.5–59.5. We must add, therefore, $\frac{2}{54}$ of 10, or 0.4, to the bottom of that interval and thus obtain 49.9 as the location of Q_1. To find Q_3, count $\frac{3}{4}$ of 200, or 150, cases from the bottom. This point is seen to lie exactly at 69.5. Hence,

$$Q = \frac{Q_3 - Q_1}{2} = \frac{69.5 - 49.9}{2} = \frac{19.6}{2} = 9.8.$$

This means that half of the cases lie within a range of 9.8 units on either side of some central point and thus tells us something of the scatter of the scores. To find the 95th percentile, we must count 0.95 of 200, or 190, cases from the bottom. We see that this point must lie in the 79.5–89.5 interval. We have 183 cases up to the bottom of this interval, so that we must add $\frac{7}{15}$ of 10, or 4.7, to the bottom of the interval and obtain the point 84.2 as the 95th percentile. We would have arrived at the same point had we counted downward 0.05 of the cases.

PROBLEMS

1. Find Q_1, Q_3, and the quartile deviation for the hourly wages in Problem 4, Section 38.

2. Find the range, quartile deviation, and the 80th percentile for the following measurements:

17	22	20	23	20
21	18	16	19	21
14	17	19	17	22
19	15	21	18	23

3. Find the quartile deviation and the 20th, 30th, and 60th percentiles for the following data.

Interval	*f*
0–2.99	5
3–5.99	9
6–8.99	21
9–11.99	11
12–14.99	6

4. Find the quartile deviation and the 45th percentile for the monthly hours worked overtime by using the data of Problem 2, Section 41.

5. Find the 10th, 20th, and 70th percentiles for the trade skill test scores of Problem 3, Section 41.

6. Find the quartile deviation of the ages of the male population in the United States in 1950 as given by the data in Problem 10, Section 41.

47. Index Numbers. In the preceding sections considerable attention has been given to the description of statistical data in terms of central tendency and dispersion. In many cases, further analysis may be made by reducing the numbers to other numbers which are purely relative, so that comparisons may be made quite easily. Among the statistical devices used for this purpose are *index numbers.* These numbers are simple ratios of one quantity to another, called the *base,* and are widely used in the study of prices, production, and other similar phases of business where variation or change is involved. Usually, the index corresponding to the base is taken to be 100, and each computed ratio is multiplied by 100, to avoid decimals.

TABLE 3

PASSENGER-CAR REGISTRATIONS IN THE UNITED STATES, 1932–1951

Year	*Passenger Cars, millions*	*Index* (1939 = 100)	*Year*	*Passenger Cars, millions*	*Index* (1939 = 100)
1932	21	81	1942	28	108
1933	21	81	1943	26	100
1934	22	85	1944	26	100
1935	23	89	1945	26	100
1936	24	92	1946	28	108
1937	25	96	1947	31	119
1938	25	96	1948	33	127
1939	26	100	1949	36	138
1940	27	104	1950	40	154
1941	30	115	1951	43	165

Let us consider the data of Table 3. The number of passenger-car registrations in the United States for each of the years 1932 to 1951 is given to the nearest million. We may reduce these numbers to a series of ratios by selecting one of them as a base. Thus, selecting 1939 as the base year, we write the index number for that year as 100. Then, for 1938, we have $(25 \div 26) \times 100 = 96$; for 1936, $(24 \div 26) \times 100 = 92$; and so forth. The index numbers obtained in this manner make possible a comparison of the number of passenger-car registrations in the United States for each year with the number of registrations for the year 1939. It can easily be seen that the number of passenger-car registrations in the United States dropped decidedly following 1941 and was below the 1941 figure for the next five years, after which there was a decided relative increase. It should be pointed out that the selection of the base year is entirely arbitrary. In their simplest form, therefore, index numbers may be calculated from a formula of the form

$$\text{Index Number} = \frac{\text{Quantity for Given Year}}{\text{Quantity for Base Year}} \times 100.$$

PROBLEMS

1. Use the year 1949 as the base year, and compute index numbers for the average weekly earnings in manufacturing industries for the years 1943 to 1952 given in the following table:*

Year	*Average Weekly Earnings*	*Year*	*Average Weekly Earning*
1943	$43.14	1948	54.14
1944	46.08	1949	54.92
1945	44.39	1950	59.33
1946	43.82	1951	64.71
1947	49.97	1952	67.97

* Source: *Business Statistics*, U.S. Department of Commerce, 1953.

2. Use 1947 as a base year, and prepare index numbers for the following average weekly earnings of employees of general merchandise stores for the years 1947 to 1952:

Year	*Earnings*
1947	$30.96
1948	33.31
1949	34.87
1950	35.95
1951	37.75
1952	38.41

3. The following table* gives the average hourly earnings in the automobile industry for the years 1943 to 1952. Compute index numbers, using 100 for the year 1949.

Year	*Earnings*	*Year*	*Earnings*
1943	$1.24	1948	$1.61
1944	1.27	1949	1.70
1945	1.26	1950	1.78
1946	1.34	1951	1.91
1947	1.47	1952	2.05

* Source: *Business Statistics,* U.S. Department of Commerce, 1953.

4. The following table* gives the average monthly anthracite coal production in millions of short tons for the years 1941 to 1952. Use 1941 as a base year, and calculate index numbers.

Year	*Production*	*Year*	*Production*
1941	4.70	1947	4.77
1942	5.03	1948	4.76
1943	5.05	1949	3.56
1944	5.31	1950	3.67
1945	4.58	1951	3.56
1946	5.04	1952	3.33

* Source: *Business Statistics,* U.S. Department of Commerce, 1953.

Exercises

1. The following table gives the distribution of the retail prices per pound of a certain cut of Grade A beef. Draw a histogram of the distribution, and compute the mean, median, modal group, Q_1, and Q_3. (Assume class intervals of $0.795–$0.835, $0.835–$0.875, and so forth.)

Price per Pound	*Frequency*	*Price per Pound*	*Frequency*
$0.80–$0.83	3	$1.08–$1.11	56
0.84– 0.87	12	1.12– 1.15	38
0.88– 0.91	25	1.16– 1.19	17
0.92– 0.95	58	1.20– 1.23	11
0.96– 0.99	71	1.24– 1.27	7
1.00– 1.03	78	1.28– 1.31	2
1.04– 1.07	62		

2. The lengths of a sample of 50 connecting pins were measured to the nearest tenth of a centimeter. The results are shown in the following distribution.

DISTRIBUTION OF LENGTHS OF 50 CONNECTING PINS

Length, Cm	*Class Mark*	*Frequency*
2.45–2.55	2.5	2
2.55–2.65	2.6	3
2.65–2.75	2.7	6
2.75–2.85	2.8	7
2.85–2.95	2.9	8
2.95–3.05	3.0	12
3.05–3.15	3.1	6
3.15–3.25	3.2	3
3.25–3.35	3.3	2
3.35–3.45	3.4	1

(*a*) Draw the histogram for these data. (*b*) Find the average (mean) length of the connecting pins measured. (*c*) What length is exceeded by $\frac{1}{10}$ of the pins? (*d*) What is the average deviation of the measurements? (*e*) What is the median length? (*f*) What is the modal group?

3. The lengths of 500 ears of corn were measured, and the data arranged as follows:

Length, in.	4	5	6	7	8	9	10	11	12	13
Number of ears	3	14	55	82	106	102	90	39	7	2

Find (*a*) the mean, (*b*) the mode.

4. The following table shows the number of students out of 375 making a test grade not higher than a given score. Find the median and quartiles.

Grade not higher than	50	55	60	65	70	75	80	85	90	95
Number of students	8	25	97	176	249	312	343	361	369	375

5. A student carries 15 hr of classwork per week and makes the following grades. If the subjects are counted equally hour for hour, what would be his average grade?

Course	*Hours per Week*	*Grade*
Mathematics	5	94
English	3	88
Chemistry	5	78
Language	2	82

6. Tabulate the following prices per dozen oranges into two frequency tables, using 5¢ and 10¢, respectively, as class intervals:

47¢	63¢	54¢	50¢	55¢	46¢	49¢	57¢	44¢	66¢	38¢	52¢	43¢	60¢	50¢
54	38	66	55	47	41	56	44	54	37	59	43	45	38	56
41	52	35	50	40	70	36	34	79	80	74	33	68	54	63
51	56	74	44	69	54	54	41	52	49	43	72	54	40	52
51	54	71	49	55	63	47	60	54	73	45	73	65	62	55
60	45	57	52	62	54	51	49	57	31	48	61	57	62	51

Find for each table (*a*) the mean, (*b*) the median, (*c*) the modal group, (*d*) the average deviation.

7. A department store manager reported the following number of employees for the period 1935 to 1954:

Year	*Number*	*Year*	*Number*	*Year*	*Number*	*Year*	*Number*
1935	482	1940	390	1945	487	1950	487
1936	306	1941	404	1946	525	1951	421
1937	310	1942	462	1947	543	1952	386
1938	323	1943	458	1948	580	1953	479
1939	327	1944	497	1949	523	1954	624

Analyze the report, and prepare a summary of your findings.

8. Twenty freshman students received the following grades in mathematics and chemistry:

Student	*Mathematics Grade*	*Chemistry Grade*	*Student*	*Mathematics Grade*	*Chemistry Grade*
A	81	88	K	74	75
B	76	80	L	65	62
C	98	95	M	84	83
D	66	71	N	73	78
E	73	70	O	87	82
F	56	63	P	70	68
G	91	80	Q	73	79
H	90	94	R	47	61
I	82	78	S	83	79
J	79	76	T	52	58

Compute the mean and standard deviation for each set of grades.

9. Compute the standard deviation of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and compare it with that obtained for the numbers 10, 20, 30, 40, 50, 60, 70, 80, 90, 100.

(*a*) What appears to be the effect on the standard deviation when a series of numbers is multiplied or divided by a constant?

(*b*) If σ_x is the standard deviation of a set of X values, prove by using the definition of standard deviation that $\sigma_{ax} = a\sigma_x$, where a is a constant. What economical advantage is made possible by this relation when the worker is dealing with large numbers?

10. Compare the standard deviation of the nunbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 with that obtained from 11, 12, 13, 14, 15, 16, 17, 18, 19, 20.

(*a*) What appears to be the effect on the standard deviation when the same constant is added to or subtracted from each number in the data?

(*b*) Prove that $\sigma_{x+a} = \sigma_x$.

Self-Test

Time: 50 Minutes

(Score 10 points for each problem correctly solved)

1. The grades for a class of 20 students in a course in business administration were: 78, 82, 76, 85, 66, 94, 96, 87, 79, 85, 95, 71, 75, 80, 81, 64, 83, 72, 75, 76. Find the mean and standard deviation.

2. The ages of 20 students attending a class were as follows: 17, 19, 18, 18, 23, 17, 18, 24, 17, 22, 19, 18, 18, 26, 18, 21, 18, 17, 18, 18. Find the mode.

3. The weights of 20 students were as follows: 145, 132, 167, 110, 184, 135, 137, 154, 163, 127, 153, 147, 139, 109, 113, 124, 138, 171, 133, 127. Find the median.

4. Find by the short method the mean and standard deviation for the following ticket prices:

Price	*Number of Tickets*
$0.90–$1.00	1
0.80– 0.90	3
0.70– 0.80	6
0.60– 0.70	10
0.50– 0.60	15
0.40– 0.50	11
0.30– 0.40	4

(Interpret intervals as $0.30–$0.399, $0.40–$0.499, and so forth.)

5. Find the third quartile for the following distribution of 100 scores in a bowling tournament:

Score	*Frequency*	*Score*	*Frequency*
250–259	3	200–209	15
240–249	7	190–199	11
230–239	12	180–189	5
220–229	16	170–179	4
210–219	26	160–169	1

(Interpret intervals as 159.5–169.5, 169.5–179.5, and so forth.)

6. From a high altitude 20 bombs were dropped over a precision target with the following results in accuracy:

Distance from Target, Ft	*Number of Bombs*
1,000	4
1,500	8
2,000	5
2,500	3
	20

Find the average number of feet by which the target was missed.

7. Prices of eggs in a certain community averaged for the period 1945 to 1954 as follows:

Year	*Price per Dozen, Cents*	*Year*	*Price per Dozen, Cents*
1945	45	1950	52
1946	43	1951	54
1947	38	1952	55
1948	46	1953	67
1949	50	1954	79

Use 1947 as a base year, and compute an index number for the average price of eggs for the year 1952.

8. If $Q_1 = 60$ and $Q_3 = 80$, what is the value of the semi-interquartile range?

9. Which would be the best measure to use—mean, median, or mode—in determining the height above the floor to build a make of kitchen range?

10. If $B =$ bottom of the interval, $C =$ number of cases in the interval, $n =$ "number of cases to go" to reach the median, and $L =$ size of the interval, write a formula by which the median may be found.

Chapter IV

PERCENTAGE AND BUSINESS APPLICATIONS

Introduction. When a businessman speaks in terms of *percentage*, he uses one of the most important applications of decimals that can be found in the business world. In fact, in almost all problems dealing with interest, discount, taxation, insurance, profit and loss, commission, and other such fields, comparisons of quantities are stated in terms of percentage. It is highly important, therefore, that the student of business thoroughly understand the meaning of percentage and that he have complete mastery of the simple mathematics underlying the uses of percentage in business and industrial fields.

48. Meaning of Percentage. In percentage a quantity is thought of as being made up of 100 equal parts, and any number of these equal parts is referred to as a certain *per cent* of the quantity. For example, when we say a salesman is to receive 3 per cent (3%) commission on selling goods, we mean that he is to receive $3 on every $100 worth of goods sold. For selling $220 worth of goods, he would receive $\frac{3}{100}$ of $220, or 0.03 times $220, which equals $6.60. It is apparent that the simple task of multiplying 0.03 times $220 is equivalent to dividing $220 into 100 equal parts of $2.20 each and then taking 3 of these equal parts for a total of $6.60. The idea that percentage refers to the number of hundredths is fundamental to a clear understanding of the meaning of percentage. The following statements will serve as further illustrations:

6 per cent (6%) of $500 means: $\frac{6}{100}$ of $500, or 0.06 × $500, which equals $30.

75 per cent (75%) of 120 cartons means: $\frac{75}{100}$, or $\frac{3}{4}$, of 120 cartons, or 0.75 × 120 cartons, which equals 90 cartons.

200 per cent (200%) of \$135 means: $\frac{200}{100}$ of \$135, or 2 × \$135, which equals \$270.

$3\frac{1}{2}$ per cent ($3\frac{1}{2}\%$) of \$50 means: $3\frac{1}{2}/100$, or $\frac{7}{200}$ of \$50, or 0.035 × \$50, which equals \$1.75.

The number on which the percentage is computed is called the *base.* The *rate* is the number of hundredths. The *percentage* is the product obtained by taking a certain per cent of the base. Thus, in the last statement of the foregoing examples, \$50 is the base, $3\frac{1}{2}\%$ is the rate, and \$1.75 is the percentage.

Since percentage problems involve common fractions and decimals, it is desirable that the student become skillful (1) in the operations involved in changing common fractions into equivalent decimals and per cents and (2) in the process of changing per cents to common fractions and decimals. These operations will now be considered in preparation for the uses of percentage to be discussed later.*

(1) Changing Common Fractions into Decimals and Per Cents: In order to change a common fraction to a per cent, our problem is to express the common fraction in terms of *hundredths,* for that is the meaning of per cent. This may be done simply by dividing the numerator by the denominator. The decimal quotient can then be written as a per cent by moving the decimal point two places to the right and affixing the per cent sign.

Illustrations:

1. Change $\frac{3}{8}$ to a per cent.

Divide the numerator by the denominator. The quotient 0.375 (three hundred seventy-five thousandths) is equivalent to "thirty-seven and one-half *hundredths*"—that is, "thirty-seven and one-half *per cent.*" Hence, $\frac{3}{8} = 0.375 = 37.5\%$, or $37\frac{1}{2}\%$.

$$\begin{array}{r} 0.375 \\ 8\overline{)3.000} \\ \underline{2\,4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \end{array}$$

2. Write 0.3547 as a per cent.

Move the decimal point two places to the right: 0.3547 = 35.47%, or $35\frac{47}{100}\%$.

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* In this chapter uniformity of answers may be obtained by rounding off percentage decimal equivalents correct to the fourth decimal place.

3. Write 2.06 as a per cent.

Move the decimal point two places to the right: 2.06 = 206%.

4. Express the quotient of 34 ÷ 153 as a per cent.

Since answers in this chapter are rounded off to the fourth decimal place, the quotient, as a per cent, is written 22.22%.

$$\begin{array}{r}
0.22222\\
153\overline{)34.00000}\\
\underline{30\,6}\\
3\,40\\
\underline{3\,06}\\
340\\
\underline{306}\\
340\\
\underline{306}\\
340\\
\underline{306}\\
34
\end{array}$$

5. Round off 0.683582 to the fourth decimal place, and express as per cent.

The rounded-off decimal is 0.6836, and therefore the per cent is 68.36%.

PROBLEMS

1. Write the following decimals as per cents:

(*a*) 0.06,
(*b*) 0.25
(*c*) $0.12\frac{1}{2}$,
(*d*) $0.33\frac{1}{3}$,
(*e*) $0.16\frac{2}{3}$,
(*f*) 0.65,
(*g*) 0.045,
(*h*) 0.0625,
(*i*) 1.25,
(*j*) 2.50,
(*k*) 5.5,
(*l*) 24.5,
(*m*) $0.11\frac{1}{9}$,
(*n*) $0.62\frac{1}{2}$,
(*o*) $0.66\frac{2}{3}$,
(*p*) $0.14\frac{2}{7}$,
(*q*) 1,
(*r*) 2,
(*s*) 10.0,
(*t*) 1000.,
(*u*) 0.07638,
(*v*) 0.53751,
(*w*) 0.92357,
(*x*) 0.41209.

2. Write the following common fractions as decimals and as per cents:

(*a*) $\frac{1}{4}$,
(*b*) $\frac{1}{2}$,
(*c*) $\frac{1}{3}$,
(*d*) $\frac{2}{3}$,
(*e*) $\frac{1}{5}$,
(*f*) $\frac{1}{7}$,
(*g*) $\frac{1}{6}$,
(*h*) $\frac{1}{8}$,
(*i*) $\frac{1}{9}$,
(*j*) $\frac{1}{10}$,
(*k*) $\frac{1}{11}$,
(*l*) $\frac{1}{16}$,
(*m*) $\frac{5}{8}$,
(*n*) $\frac{16}{6}$,
(*o*) $\frac{7}{8}$,
(*p*) $\frac{5}{32}$,
(*q*) $\frac{9}{25}$,
(*r*) $\frac{240}{300}$,
(*s*) $\frac{1662}{850}$,
(*t*) $\frac{1275}{100}$,
(*u*) $\frac{21}{8}$,
(*v*) $\frac{99}{7}$,
(*w*) $\frac{93}{12}$,
(*x*) $\frac{22}{7}$.

(2) **Changing Per Cents to Decimals and Common Fractions:** To change a per cent to a decimal, we simply place the decimal point so as to indicate the proper number of *hundredths.* Thus, 6% = 0.06; 45% = 0.45; 12.5% = 0.125; 225% = 2.25, and so forth. It is readily apparent that in changing from a per cent to a decimal we merely drop the per cent sign and move the decimal point two places to the left.

In order to change a per cent to a common fraction, we may first write it as a decimal and then express the decimal as a common fraction by dropping the decimal point and dividing by the proper denominator. The resulting common fraction may then be reduced to lowest terms. Thus:

$$\begin{aligned}
6\% &= 0.06 = \tfrac{6}{100} = \tfrac{3}{50}.\\
65\% &= 0.65 = \tfrac{65}{100} = \tfrac{13}{20}.\\
12.5\% &= 0.125 = \tfrac{125}{1000} = \tfrac{1}{8}.\\
300\% &= 3.00.\\
1\% &= 0.01 = \tfrac{1}{100}.\\
0.1\% &= 0.001 = \tfrac{1}{1000}.\\
33\frac{1}{3}\,\% &= \frac{33\frac{1}{3}}{100} = \frac{\frac{100}{3}}{100} = \frac{100}{3} \times \frac{1}{100} = \frac{100}{300} = \frac{1}{3}.
\end{aligned}$$

PROBLEMS

1. Write the following per cents as decimals and as common fractions:

(a) 3%,
(b) $4\frac{1}{2}$%,
(c) $8\frac{1}{3}$%,
(d) $37\frac{1}{2}$%,
(e) $62\frac{1}{2}$%,
(f) 42%,
(g) 95%,
(h) 125%,
(i) 225%,
(j) 400%,
(k) $\frac{1}{2}$%,
(l) $\frac{3}{4}$%,
(m) 435%,
(n) 600%,
(o) 1,000%,
(p) 6.5%,
(q) 0.5%,
(r) $\frac{2}{7}$%,
(s) 0.04%,
(t) 0.026%.

2. By changing the form of the figures, fill in the blanks as indicated.

	Decimal	*Common Fraction*	*Per Cent*		*Decimal*	*Common Fraction*	*Per Cent*
(a)	______	$\frac{3}{4}$	______	(g)	______	$\frac{39}{8}$	______
(b)	______	______	36%	(h)	26.4	______	______
(c)	0.025	______	______	(i)	______	______	260%
(d)	______	______	0.06%	(j)	______	$\frac{13}{8}$	______
(e)	______	$\frac{5}{7}$	______	(k)	99.5	______	______
(f)	______	______	5.6%	(l)	0.625	______	______

49. Types of Percentage Problems. Percentage problems may be discussed under three headings, according to what is required in the problem. As we have seen, the three elements which enter into percentage are the *base*, the *rate*, and the *percentage*. In practice we usually know two of these elements and are required to find the third. The three types will be considered separately.

(1) **To Find a Per Cent of a Number:** Find 5% of 640.

Multiply by 0.05,
$$\begin{array}{rl} 640 & \text{(Base)} \\ .05 & \text{(Rate)} \\ \hline 32.00 & \text{(Percentage)} \end{array}$$

Multiply by $\frac{1}{20}$.
$$\tfrac{1}{20} \times 640 = \tfrac{640}{20} = 32.$$

We see that *percentage equals base times rate* and that the work may be carried out in terms of either decimal or common fractions. In either case, the operation is expressed by multiplying the base times the rate. In symbols, we may write the equation,

$$P = B \times R. \qquad (a)$$

Illustration: A college student council allotted 23% of a budget of $3,690 to publications and 40% to athletics. How much was allotted to these two items?

$$\begin{aligned} B &= 3{,}690, \quad R = 0.23, \\ P &= B \times R \\ &= 3{,}690 \times 0.23 \\ &= \$848.70 \text{ for publications.} \end{aligned}$$

$$\begin{aligned} B &= 3{,}690, \quad R = 0.40, \\ P &= B \times R \\ &= 3{,}690 \times 0.40 \\ &= \$1{,}476 \text{ for athletics.} \end{aligned}$$

PROBLEMS

1. Memorize the following table:

$\frac{1}{2} = 50\%$,
$\frac{1}{3} = 33\frac{1}{3}\%$,
$\frac{2}{3} = 66\frac{2}{3}\%$,
$\frac{1}{4} = 25\%$,
$\frac{3}{4} = 75\%$,
$\frac{1}{5} = 20\%$,
$\frac{2}{5} = 40\%$,
$\frac{3}{5} = 60\%$,
$\frac{4}{5} = 80\%$,
$\frac{1}{6} = 16\frac{2}{3}\%$,
$\frac{5}{6} = 83\frac{1}{3}\%$,
$\frac{1}{8} = 12\frac{1}{2}\%$,
$\frac{3}{8} = 37\frac{1}{2}\%$,
$\frac{5}{8} = 62\frac{1}{2}\%$,
$\frac{7}{8} = 87\frac{1}{2}\%$,
$\frac{1}{12} = 8\frac{1}{3}\%$,
$\frac{5}{12} = 41\frac{2}{3}\%$,
$\frac{1}{16} = 6\frac{1}{4}\%$,
$\frac{1}{20} = 5\%$,
$\frac{1}{25} = 4\%$.

2. Find the following thus: Look at "$12\frac{1}{2}\%$," think "$\frac{1}{8}$." Then, $\frac{1}{8} \times 24 = 3$.

(*a*) $12\frac{1}{2}\%$ of 24,
(*b*) $37\frac{1}{2}\%$ of 80,
(*c*) 4% of \$75,
(*d*) 20% of \$155,
(*e*) 75% of 32 yd,
(*f*) $83\frac{1}{3}\%$ of 42 gal,
(*g*) 40% of 80 ft,
(*h*) 5% of 40 shares,
(*i*) $66\frac{2}{3}\%$ of 18 cartons,
(*j*) $6\frac{1}{4}\%$ of 64 lb,
(*k*) 50% of \$250,
(*l*) $33\frac{1}{3}\%$ of 96 rolls,
(*m*) 25% of 480 chairs,
(*n*) $41\frac{2}{3}\%$ of 60 in.
(*o*) $87\frac{1}{2}\%$ of \$880,
(*p*) $62\frac{1}{2}\%$ of \$880,
(*q*) $83\frac{1}{3}\%$ of 24,
(*r*) $16\frac{2}{3}\%$ of 486,
(*s*) 80% of \$175,
(*t*) 75% of 140 employees.

3. Find the following thus: $P = B \times R$, $P = 300 \times 0.06 = 18$.

(*a*) 6% of \$300,
(*b*) 40% of 30 days,
(*c*) 15% of \$4,000,
(*d*) 22% of 44,
(*e*) 5% of 350,
(*f*) 8% of 362,
(*g*) 4% of 85 days,
(*h*) $4\frac{1}{2}\%$ of \$850,
(*i*) 5% of \$432,
(*j*) 6% of \$764.25,
(*k*) 7% of \$23,750,
(*l*) 8% of 456.7,
(*m*) $2\frac{1}{2}\%$ of \$32,754.53,
(*n*) $3\frac{1}{4}\%$ of 400 bonds.

4. A real estate agent sells a house for \$19,000 and charges 2% commission. Find his commission and the net proceeds. (The net proceeds is the sum left after commission and other expenses have been paid.)

5. A salesman sold \$76,000 worth of steel products in a year at a commission of 7%. If his expenses for the year were \$1,800, how much did he have left?

6. A discount of 25% was allowed on a radio marked to sell for \$165. What was the purchase price?

(2) To Find What Per Cent One Number Is of Another: What per cent of \$320 is \$19.20?

Here the rate is the missing item in the relationship $P = B \times R$. Solving for R, we have

$$R = P \div B, \tag{b}$$

$B = 320$, $P = 19.20$. Substituting these values,

$$R = 19.20 \div 320 = 0.06 = 6\%.$$

We see that *rate equals percentage divided by base.*

Problems

1. What per cent of:

(*a*) 20 is 10?
(*b*) 15 is 3?
(*c*) 35 is 7?
(*d*) 40 lb is 10 lb?
(*e*) $1,200 is $72?
(*f*) $24.50 is $49?
(*g*) $4,800 is $168?
(*h*) 60 in. is 12 in.?
(*i*) 80 days is 10 days?
(*j*) $4\frac{4}{5}$ oz is $1\frac{1}{5}$ oz?
(*k*) 20 is 40?
(*l*) $60 is $90?
(*m*) 254 is 86?
(*n*) $321 is $22.47?
(*o*) $500 is $14.50?

2. What per cent of:

(*a*) 1 dollar is 1 dime?
(*b*) $1\frac{1}{2}$ yd is $\frac{1}{2}$ yd?
(*c*) 32 qt is 6 qt?
(*d*) $\frac{8}{9}$ is $\frac{5}{9}$?
(*e*) 0.095 is 0.038?
(*f*) $4 is 15 cents?
(*g*) 0.26 is 0.0052?
(*h*) 22 is 1.65?
(*i*) 340 is 15.30?
(*j*) 86.3 is .1726?
(*k*) $\frac{7}{11}$ is $\frac{3}{22}$?
(*l*) $24\frac{3}{4}$ is $5\frac{1}{2}$?
(*m*) $67,392.60 is $5,615.05?
(*n*) 46,358 is 32,451?
(*o*) 21,690 is 28,920?

3. An income of $137.50 was realized on an investment of $2,500. What per cent was made on the investment?

4. In a certain business college, 40 students are enrolled in bookkeeping, 90 are enrolled in typewriting, and 60 are enrolled in accounting. What per cent of the total number is enrolled in each course?

5. A $60 suit is sold for $49.95. What per cent reduction is allowed?

6. Over a four-month period the following stock quotations changes were noted. Find the per cent of "advance" or "drop" for each.

Stock	*June 1*	*Oct. 1*	*Per Cent Change*	*Stock*	*June 1*	*Oct. 1*	*Per Cent Change*
ATT	73	82	______	MCC	$116\frac{5}{8}$	$122\frac{1}{4}$	______
BD	85	$72\frac{1}{2}$	______	LDD	$4\frac{3}{4}$	$5\frac{1}{4}$	______
CC	106	102	______	PMD	$52\frac{1}{2}$	$49\frac{1}{4}$	______
DM	67	$76\frac{1}{2}$	______	OFA	190	195	______
RCA	$94\frac{1}{4}$	$97\frac{3}{4}$	______	FHB	112	$116\frac{1}{4}$	______

7. A college bookstore sold 235 mathematics books at \$6 each and made a profit of \$169.20. This profit was what per cent of the total amount received for the books?

8. In 1939, the per-capita public debt of the United States was \$309, and in 1952 it was \$1,650. What was the per cent of increase in the per-capita public debt for this period?

9. In 1949, an average of 49,400 short tons of zinc per month was mined in the United States and Alaska. In 1952, the corresponding output was 55,100 short tons. What per cent increase is represented?

10. An investor bought common stock in a corporation at \$105 per share and received dividends of \$3.40, \$1.20, and \$1.70 during the year. What was his rate of return?

11. Mr. Morrison bought preferred stock in the High Voltage Corporation at \$206 per share and received during the year dividends totaling \$9.27. What was his rate of return?

(3) To Find a Number When a Given Per Cent of It Is Known: If 4% of a number is 12, what is the number?

Here the *base* is the missing item in the formula $P = B \times R$. Solving for B, we have

$$B = P \div R, \tag{c}$$

$P = 12$, $R = 0.04$. Substituting these values,

$$B = 12 \div 0.04 = 300.$$

We see that *base equals percentage divided by rate.*

PROBLEMS

1. 3% of what number is 3?
2. 50% of what number is 16?
3. 25% of what number is 10?
4. 6% of what number is 3?
5. 20% of what number is 7?
6. 5% of what amount is \$3?
7. 7% of what amount is \$4.55?
8. 6% of what amount is \$17.52?
9. $3\frac{1}{2}$% of what amount is \$17.50?
10. $5\frac{1}{4}$% of what amount is \$126?

11. The net profit of a haberdashery shop for the year is $5,000, or $6\frac{1}{4}\%$ of the total sales. What were the total sales?

12. A certain store had its yearly rent increased 10%, or $640, over the previous year. How much did it formerly pay?

13. The following news item appeared in a Cleveland newspaper:

> New York, Jan. 3—(AP)—An all-time high of $250,000,000 in phonograph record sales in 1954 was forecast today by Frank M. Folsom, president of Radio Corporation of America. He said this would represent a gain of 10% over 1953 which in turn were 12% ahead of those of 1952.

Using these data, determine the 1952 and 1953 phonograph record sales to the nearest million dollars.

14. Thomas Baker received a withholding statement from his employer showing that $72 had been withheld from his yearly salary for Federal old age and survivors insurance. If the deduction was at the rate of 2%, what amount of his yearly salary was taxable?

15. A withholding statement showed that Fred Smith had $486 withheld during the year for Federal income tax. If the applicable tax rate was 12%, what was his yearly salary?

16. Mr Brown bought 200 shares of 4% preferred stock at $98, 100 shares of 5% preferred stock at $108, and a $5,000 $4\frac{1}{2}\%$ mortgage at par. What was his average rate of income if $100 was the par value of each of the shares?

50. Commission. Many applications of percentage may be found in commission, trade discounts, insurance, taxation, profit and loss, and in other commercial fields where comparisons of quantities are involved. Interest and bank discount are such important applications that they will be considered in a separate chapter. The remainder of the present chapter will be devoted to several fields of application, in order that the student may become familiar with the types of problems encountered.

Commission is a certain per cent of the amount of money involved in a business transaction which is carried out by an agent, or one who transacts business for another. A *commission merchant*, for example, gets a certain per cent of the amount collected; a *broker* gets a certain per cent of the cost or the selling price.

The *net proceeds* is the amount left after the commission and other expenses have been paid.

Illustrations:

1. Mr. J. W. Blythe, salesman for the Star Packing Company, receives a commission of 12% on sales amounting to $876.30. Find his commission.

Amount of sales,	$876.30
	0.12
	175260
	87630
	$105.1560

The commission is seen to be $105.16.

2. A wholesale firm sold 80 bbl of wine at $72 per barrel, on a commission of 8%. The freight charges were $215.60, and trucking expenses amounted to $47.20. Find the commission and the net proceeds of the sale.

Amount of sale, 80 bbl at $72		$5,760.00
Commission, 8% of $5,760	$460.80	
Freight	215.60	
Trucking	47.20	723.60
Net Proceeds		$5,036.40

PROBLEMS

1. A lawyer charged his client 5% for collecting a bill amounting to $465. Find the lawyer's fee and the amount he remitted.

2. For selling 100 copies of the *Readers Review* a boy received a baseball glove worth $2.50. If the copies sold for 25¢ each, what was his actual commission rate? How much did he receive per copy?

3. A salesman agreed to sell goods on the following basis: $100 per month plus 20% on all sold above $60,000 during the year. What would be an equivalent straight commission rate for selling $62,700 worth of goods in one year?

4. A contact man for the Stetson Steel Company gets 2% commission from his company on all government contracts arranged for. If he pays his own expenses, which amount to $3,620, what is his net income on a total contract amounting to $1,987,500?

5. A stock broker charges $\frac{1}{2}\%$ for selling stocks. If his brokerage on a certain sale amounted to $181.20, what was the amount of the sale?

6. A lawyer gives a client $815.24 after taking out 11% for his services in collecting a claim. What was the amount of the claim?

7. Find the net proceeds and other missing information:

Gross Proceeds	*Freight and Cartage*	*Collection and Guaranty*	*Commission*	*Other Charges*	*Total Charges*	*Net Proceeds*
$ 400	$ 15.20	2% = $8.00	$3\frac{1}{2}\%$ = $14.00	$ 8.65	$45.85	$354.15
3,000	84.00	2% = ____	$3\frac{1}{2}\%$ = ____	7.90	____	____
2,150	40.00	2% = ____	$3\frac{1}{2}\%$ = ____	4.00	____	____
6,010	128.50	$1\frac{1}{2}\%$ = ____	$3\frac{1}{2}\%$ = ____	27.00	____	____

8. A lawyer received a claim of $260 for collection. He collected 85% of the claim and charged 15% of the amount collected for his services. What did his client receive?

9. A commission broker received an average commission of $640 per month for the first six months of the year. His July and August earnings were $300 and $260, respectively. What were his average earnings per month to date? What percentage decrease in average monthly commission did he experience during these two months over the average monthly commission for the first six months?

10. A salesman for the Young Kitchen Company is paid $3,600 a year plus a commission at the end of the year calculated as follows: $\frac{1}{2}\%$ on sales up to $200,000, $\frac{3}{4}\%$ on sales from $200,000 to $400,000, and 1% on sales over $400,000. What are his yearly earnings earnings on sales for the year totaling $527,450?

51. Cash and Trade Discounts. It is common business practice for a seller to offer the buyer the privilege of deducting a certain per cent of the list price of goods, or the amount of a bill, from the total amount of the bill if payment is made before the expiration of a stated time. The amount deducted is known as a *cash discount.* Thus, a bill for $200 payable within 30 days but subject to a discount of 3% if paid within 10 days means that 3% of the *net,* or full, amount due may be subtracted from the amount of the bill, leaving \$200 − (0.03 × \$200) = \$200 − \$6 = \$194 as the amount to be paid if remittance is made within the 10 days. The terms of such a discount are often indicated symbolically. Thus, "3/10, *n*/30" appearing on an invoice or a bill would mean 3% off for payment within 10 days, and net, or full, amount payable if paid after 10 days but within 30 days.

To allow dealers to make adjustments in list prices published in catalogs, manufacturers and wholesalers allow *trade discounts*. Frequently, a chain of *successive* discounts are offered to enable dealers to adjust to changing prices. The first discount is taken off the list price, the second off the first net price after the first discount has been taken, and so on. The computation of the net amount of a bill subject to successive discounts may be illustrated as follows:

Illustration: A high-fidelity speaker is listed at \$80 in an electrical supply catalog. It is sold to a retailer who received trade discounts of 20%, 10%, and 5%. Find the net price of the speaker.

$$\begin{array}{rrl}
 & \$80.00 & = \text{list price} \\
20\% \text{ of } \$80 = & \underline{16.00} & = 20\% \text{ trade discount} \\
 & \$64.00 & = \text{first net price} \\
10\% \text{ of } \$64 = & \underline{6.40} & = 10\% \text{ trade discount} \\
 & \$57.60 & = \text{second net price} \\
5\% \text{ of } \$57.60 = & \underline{2.88} & = 5\% \text{ trade discount} \\
 & \$54.72 & = \text{final net price}
\end{array}$$

Two successive discounts may be calculated as one single discount by taking as the single equivalent rate the sum of the two successive discount rates minus the product of these rates. The proof for the formula is as follows:

Let P = the original sum, d_1 = the first discount rate, and d_2 = the second discount rate. Then,

$$\begin{aligned}
Pd_1 &= \text{first discount,} \\
P - Pd_1 &= \text{amount left after first discount,} \\
(P - Pd_1)d_2 &= \text{second discount,} \\
(P - Pd_1) - (P - Pd_1)d_2 &= \text{amount left after second discount.}
\end{aligned}$$

Removing parentheses, we may write this amount as $P - Pd_1 - Pd_2 + Pd_1d_2$, and then as $P - P(d_1 + d_2 - d_1d_2)$. Let

$$D = d_1 + d_2 - d_1d_2.$$

Then $P - PD$ = amount left after two successive discounts.

This is an expression for a single discount at the rate D. Hence, *a single discount equal to two successive discounts may be found by taking as the rate the sum of the two successive discount rates minus the product of those rates.* In a similar fashion, formulas may be derived for determining equivalent single discount rates for three or more successive discounts (see Problem 9 in following list).

Illustration: Find the amount due on a bill of \$240 after two successive discounts of 15% and 10% have been allowed.

$0.15 + 0.10 = 0.25$ (Sum of the rates),
$0.15 \times 0.10 = 0.015$ (product of the rates),
$0.25 - 0.015 = 0.235$ (equivalent single rate),
$\$240 \times 0.235 = \$56.40$ (total discount),
$\$240 - \$56.40 = \$183.60$ (amount due after both discounts have been allowed).

Check:

$$\begin{array}{lrl} & \$240.00 & \\ 15\% \text{ of } \$240 = & 36.00 & \\ \hline & \$204.00 & \text{(Amount left after first discount)} \\ 10\% \text{ of } \$204 = & 20.40 & \\ \hline & \$183.60 & \text{(Amount left after second discount)} \end{array}$$

PROBLEMS

1. Find the net cost of articles listed at:

    (*a*) \$140, discount 20%, 10%,
    (*b*) \$230, discount 25%, 20%,
    (*c*) \$560, discount 12%, 6%,
    (*d*) \$500, discount 20%, 10%,
    (*e*) \$5,600, discount $37\frac{1}{2}$%, 25%,

(*f*) \$325, discount 6%, 4%, 2%,
(*g*) \$538.60, discount 25%, $12\frac{1}{2}$%,
(*h*) \$2,300, discount 10%, 10%.

2. What single discount is equivalent to successive discounts of 20% and 10%?

3. Goods marked \$250 were bought at 10% trade discount and 2% off for cash. If sold for \$264.60, how much was the profit? This profit was what per cent of the cost of the goods?

4. A grocer is offered a discount of 20% from one firm on a bill of goods amounting to \$2,360 and two successive discounts of 10% by another firm on the same bill. Which is the better offer, and how much better?

5. Three firms bid on the paint for a building as follows: (*a*) \$3,200, 50%, 20% off. (*b*) \$3,600, 40%, 30% off. (*c*) \$3,100, 60%, 10% off. Which is the lowest bidder?

6. What single discount is equivalent to two successive discounts of 30% and 20%?

7. The list price of an invoice is \$800, with discounts of 10%, 5%, and $2\frac{1}{2}$%. Terms: 2/10, 1/30. What amount is necessary to pay the invoice within 10 days? Within 30 days? (*Note:* 2/10 means that 2% additional discount is allowed if paid within 10 days.)

8. Compute the single discount that is equivalent to each of the following successive discounts: (*a*) 10%, 5%, (*b*) 20%, 15%, (*c*) 35%, 20%, (*d*) 15%, 5%, (*e*) 40%, 30%, (*f*) 60%, 45%.

9. Derive the formula for finding a single discount rate equivalent to three successive discount rates: $D = 1 - (1 - d_1)(1 - d_2)(1 - d_3)$.

10. Three successive discounts of 20%, 10%, and 5% are given on an article listed at \$100. (*a*) Find the cost of the article by applying the discount rates in succession. (*b*) Find the cost of the article by using the formula for single discount rate stated in Problem 9.

11. Compute the single discount that is equivalent to each of the following successive discounts: (*a*) 20%, 10%, 10%, (*b*) 30%, 20%, 15%, (*c*) 40%, 20%, 10%, (*d*) 30%, 30%, 10%, (*e*) r%, $2r$%, $3r$%, (*f*) K%, $K/2$%, $K/4$%.

12. An invoice of goods is purchased for \$28,400 after discounts of 10%, 10%, and 10% have been allowed. What was the list price? (*Hint:* Let x = list price.)

13. Smith, Adler, and Hoyle, Limited, purchases a bill of goods for \$5,100 less 15%, 9%, and 4% and sells them at discounts of 20% and 8%. Is there a gain or loss? How much?

14. Two successive discounts of $p\%$ and $q\%$ were given on a bill for S dollars. Find the net proceeds.

15. The list price of a dining-room light fixture is \$48. Trade discounts of 20% and 20% are given the dealer, and the terms of payment are 5/10, n/30. Find the net price of the fixture if the bill is paid within 10 days.

16. Find the net amount of a bill for \$220, less trade discounts of 10% and 10%, dated Oct. 15, terms 3/10, n/30, if paid on Oct. 20.

52. Taxation. The computation of taxes on real estate and personal property is another application of percentage. The assessed valuation, or the value of the property on which the tax is based, or an individual's taxable income is the *base* on which the tax (*percentage*) is calculated. The rate of taxation corresponds to the *rate* in any percentage problem.

Illustration: The following income tax data were prepared by an individual whose gross income amounted to \$6,112.

Income

1. Salaries and other compensation for personal services	\$5,600.00	
2. Dividends	380.00	
3. Interest on bank deposits, notes, mortgages, etc.	62.00	
4. Interest on corporation bonds		
5. Other income (including income from annuities, etc.)	70.00	
6. Total income in items 1 to 5		\$6,112.00

Deductions

7. Contributions paid	\$ 100.00	
8. Interest paid	14.60	
9. Taxes paid	130.80	
10. Other deductions authorized by law	10.00	
11. Total deductions in items 7 to 10		\$ 255.40

Computation of Tax

12.	Net income (item 6 minus item 11)		$5,856.60
13.	Less: Earned income credit, either (*a*) or (*b*)		
	(*a*) If item 12 is $3,000 or less, enter 10% of such item		
	(*b*) If item 12 is more than $3,000, enter 10% of item 1 or 10% of item 12, whichever is smaller, but not less than $300	$ 560.00	
14.	Credit for dependents		
15.	Personal exemption	2,000.00	$2,560.00
16.	Balance of net income taxable (item 12 minus 13, 14, 15)		$3,296.60
17.	Income tax (4% of item 16)		$ 131.86
18.	Less: Income tax paid at source on tax-free covenant bonds	$	
19.	Income tax paid to a foreign country or United States possession	$	
20.	Balance of income tax (item 17 minus items 18 and 19)		$ 131.86
21.	Surtax (10% of item 17)		$ 13.19
22.	Total tax due (item 20 plus item 21)		$ 145.05

Problems

1. Compute the total income tax for an individual whose income included a salary of $2,400, $26 from dividends, and $34.50 interest on a mortgage. His contributions to charitable organizations amounted to $85, and he was allowed $22 for taxes paid. He claimed $1,000 personal exemption and $400 credit for dependents. (Use the form in the above illustration.)

2. Mr. Stephens's city tax, at the rate of 16 mills on the dollar, is $127.94. What is the value of his property, if it is assessed at $\frac{4}{5}$ of its value? (*Note:* A mill is one-tenth of a cent.)

3. Find the tax on property assessed at (rate given in mills and cents on the dollar) (*a*) $5,000, tax levy 7 mills, (*b*) $12,400, tax levy 12 mills, (*c*) $35,000, tax levy $3\frac{1}{2}$ mills, (*d*) $362,800, tax levy 2 cents.

4. What is the tax rate (mills on the dollar) if the assessment is (*a*) $12,000, taxes $150, (*b*) $6,500, taxes $196, (*c*) $10,000, taxes $40, (*d*) $1,900, taxes $7.60.

5. Find the assessed valuation if the taxes are (*a*) $120, rate 16 mills, (*b*) $13, rate 26 mills, (*c*) $800, rate 35 mills, (*d*) $275, rate 42 mills.

6. Suppose the tax rate in a certain community is \$2.91 per \$100. Find the assessed value corresponding to each of the following taxes:

Tax Paid	*Assessed Value*
\$ 139.00	______
12,407.92	______
3,002.00	______
23,862.00	______
1,300.82	______
1,701.67	______

7. A suburban property is assessed at \$53,000. What total amount must be paid if the tax rate is \$1.426 per \$100 and an accountant's fee is 1% of the tax?

8. A state sales tax of 3% is computed on the purchase price of goods as follows: 3% of the whole number of dollars plus 1¢ for odd cents between 11¢ and 35¢ inclusive, 2¢ for amounts between 36¢ and 65¢ inclusive, and 3¢ for amounts between 66¢ and 99¢ inclusive. Compute mentally the sales tax on each of the following prices:

(*a*) \$1,
(*b*) \$4,
(*c*) \$1.60,
(*d*) \$4.29,
(*e*) 5¢,
(*f*) 89¢,
(*g*) 65¢,
(*h*) \$125,
(*i*) \$2,200,
(*j*) \$199.49,
(*k*) \$8.25,
(*l*) \$22.50,
(*m*) \$13.29,
(*n*) \$2.69,
(*o*) 6¢.

9. At certain times on certain items, the Federal government levies a retail excise tax which is computed on the retail price of the items involved. Assume a 20% excise tax and the sales tax structure in Problem 8, and compute for each of the following retail prices the corresponding cost to a purchaser:

(*a*) \$3.25,
(*b*) \$4.85,
(*c*) \$0.65,
(*d*) \$565,
(*e*) \$2.39,
(*f*) \$0.85,
(*g*) \$1.59,
(*h*) \$87.25,
(*i*) \$0.35,
(*j*) \$19.95,
(*k*) \$1.15,
(*l*) \$3.49,
(*m*) \$319.50,
(*n*) \$6.75,
(*o*) \$24.95.

53. Profit and Loss. *Profit* and *loss* are terms used to designate the monetary gains or losses incurred in the operation of a business. They are sometimes expressed in terms of per cent of *net cost* (actual amount paid for the article) but more often as a per cent of the *net selling price* (actual amount received for the article).

From the time an article is first purchased from a manufacturer or wholesaler until the time it is sold by a dealer to a customer, certain calculations, involving trade and cash discounts enjoyed by the dealer, buying expenses, selling expenses, overhead, discounts to be allowed the customer, and profit for the dealer, must be made before determination can be made of the actual price to be paid by the customer. Definitions of the various terms involved in buying and selling have been rather uniformly established in the business world; and their meanings should be clear from the following basic relationships:

Net Cost = List Price − Discounts (allowed dealer).
Gross Cost = Net Cost + Buying Expenses (transportation commission, etc.).
Marked Price = Gross Cost + Overhead + Selling Expenses + Net Profit + Discounts (allowed customer).
Selling Price = Marked Price − Discounts (allowed customer).
Gross Profit = Selling Price − Gross Cost.
Net Profit = Gross Profit − Overhead − Selling Expenses.

Illustration: A television picture tube is listed at $80 subject to a trade discount of 25%. The freight is $3, and the buying commission is $4. At what price must the tube be sold by a dealer who purchased it so as to make a net profit of 20% on the selling price if 10% of the selling price goes for overhead and 5% for selling expenses? What is the net cost? Gross cost? Marked price? Selling price? Gross profit? Net profit?

Solution: Let x = the actual selling price of the television tube. Then,

$$\begin{array}{r l}
\$80.00 & = \text{list price} \\
\underline{20.00} & = \text{trade discount } (25\% \text{ of } \$80) \\
\$60.00 & = \text{net cost to the dealer} \\
3.00 & = \text{freight} \\
\underline{4.00} & = \text{buying expenses} \\
\$67.00 & = \text{gross cost}
\end{array}$$

$0.10x$ = overhead (10% of selling price)
$0.05x$ = selling expenses (5% of selling price)
$0.20x$ = net profit (20% of selling price)

Hence, since in this illustration the marked price is the selling price to the customer, we have

$$x = \$67 + 0.10x + 0.05x + 0.20x,$$

from which we find

$$\begin{aligned} 0.65x &= \$67, \\ x &= \$103.08, \text{ selling price.} \end{aligned}$$

Our calculations show that the net cost of the picture tube is \$60. To this amount are added the buying expenses, yielding \$67 as the gross cost. To the gross cost the dealer adds overhead, selling expenses, and net profit to obtain the selling price, which, as we see, turns out to be \$103.08.

The gross profit, sometimes called *markup*, or the difference between selling price and gross cost, is found to be \$103.08 − \$67, or \$36.08, which is 53.85% markup on gross cost. The net profit is 20% of \$103.08, or \$20.62.

PROBLEMS

1. For each of the following items purchased by a dealer from a manufacturer, find the selling price and the gross profit:

Item	*Gross Cost*	*Markup on Cost*	*Net Profit on Selling Price*	*Selling Price*	*Net Profit*
(*a*)	\$ 42.50	20%	12%	———	———
(*b*)	123.80	25%	8%	———	———
(*c*)	264.90	32%	18%	———	———
(*d*)	593.00	35%	11.5%	———	———
(*e*)	85.06	26%	7.4%	———	———

2. At what price must an article costing \$32 be sold in order to make a gross profit of 20% on the selling price?

3. A desk was sold for \$52.30, giving a gross profit of 25% on the selling price. What was the gross cost of the desk?

4. An electric fan was purchased for $18.75 and was sold for $24.95. What was the per cent of gross profit on the selling price?

Complete the following table:

Item	Gross Cost	Overhead and Selling Exp. (Per Cent of Selling Price)	Gross Profit	Net Profit	Discount on Marked Price	Selling Price	Marked Price
5.	$25	10%	_____	_____	10%	_____	$40
6.	$100	10%	_____	_____	None	_____	$150
7.	$45	12½%	_____	_____	20%	$64	_____
8.	$150	9%	_____	_____	15%	$200	_____
9.	$2.20	10%	_____	$0.45	None	_____	_____
10.	$3.50	15%	_____	$0.85	None	_____	_____
11.	_____	10%	$6.00	$7.00	25%	_____	_____
12.	_____	5%	$0.17	$0.15	None	_____	_____

13. What is the percentage markup in Problem 7?

14. What is the percentage markup in Problem 8?

15. What is the per cent net profit based on the selling price in Problem 11?

16. What is the per cent net profit based on the selling price in Problem 8?

17. A dealer bought a chair from a manufacturer. The list price was $50 with a trade discount of 10%. At what price must the dealer sell the chair so as to make a net profit of 25% on the selling price, assuming overhead to be 10% of the selling price? What is the gross profit on one chair?

18. The Allan Appliance Company bought a refrigerator from a manufacturer for $225 and paid $20 shipping charges. At what price must the refrigerator be marked in order to allow a discount of 25% and still make a gross profit of 20% on the selling price?

19. The gross cost of a certain table-model television set is $189. It is marked for sale at $310 but goes to a customer at 20% off for cash. What per cent gross profit on the selling price is realized?

20. The net cost of a can of peaches is 32¢, shipping expenses are 3.2¢, and the total of overhead and selling expenses comes to

10% of the selling price. At what price should the peaches be sold to yield a 2% net profit on the selling price?

21. For convenience, merchants frequently use a *selling-price factor* which is defined by the equation

$$\text{Selling Price} = \text{Gross Cost} \times \text{Selling-price Factor.}$$

Determine the selling-price factor for each of the following:

Item	*Gross Cost*	*Per Cent Overhead on Selling Price*	*Per Cent Net Profit on Selling Price*	*Selling-price Factor*
(*a*)	$40	20%	15%	
(*b*)	$65	15%	20%	
(*c*)	$2.50	10%	5%	
(*d*)	$16	15%	30%	

Exercises

1. Change (*a*) $1.2\frac{1}{2}\%$ to a decimal, (*b*) $18\frac{3}{4}\%$ to a fraction, (*c*) $\frac{2}{15}$ to a per cent, (*d*) 1.5 to a per cent, (*e*) $r\%$ to a fraction, (*f*) 400 to a per cent.

2. If 1,500 trucks can be filled by 4,500 men in 4 hr, how many hours would it take 5,000 men to fill $66\frac{2}{3}\%$ more trucks?

3. Out of 16 homes in a certain city block, 11 contributed to a community chest drive. What per cent failed to contribute?

4. What per cent of $\frac{2}{3}$ is $\frac{3}{4}$ of $\frac{5}{8}$?

5. One dealer offers a bill of goods at a list price of $3,500 less 20%, 15%, and 5%. Another offers the same goods at a list price of $3,000 less 20%, 5%, and 5%. Which is the better offer? How much better?

6. If an article costs $1 and is marked $1.25, in order to sell the article for the cost the price must be reduced 25¢. Since 25¢ is $\frac{1}{5}$ of $1.25, the *markdown* is 20%. In the following problems the cost and the marked price are given. Compute the markdown per cent for each of the items given.

Item	*Gross Cost*	*Marked Price*	*Markdown Per Cent (to Sell at Cost)*
(*a*)	$ 0.80	$ 1.00	
(*b*)	5.00	7.00	
(*c*)	2.00	3.00	
(*d*)	0.06	0.08	
(*e*)	16.00	24.00	
(*f*)	40.00	75.00	

7. Find the net price in each of the following sales: (*a*) $400 less 10%, 5%, 5%; (*b*) $470 less 20%, 10%, 5%; (*c*) $1,500 less 4%, 1%, 1%; (*d*) $800 less 40%, 15%, 10%.

8. What single discount is equivalent to successive discounts of 20% and 20%?

9. In a certain time study it was found that the time for performing certain operations should be changed as follows:

(*a*) For boring pinholes in a connecting rod, from 1.03 time minutes to 0.94.

(*b*) For centering ends of camshaft, from 0.58 time minutes to 0.40.

(*c*) For filing burrs on connecting rod, from 0.63 time minutes to 0.72.

(*d*) For milling ends of water pump, from 0.87 time minutes to 0.71.

Find the percentage change in time for each case.

10. A department store sales tabulation for a given month in two successive years ran as follows:

Dept.	*This Month This Year*	*This Month Last Year*
A	$3,012.14	$2,947.87
B	1,735.78	1,894.65
C	4,111.10	3,858.05

(*a*) Find the increase or the decrease in monthly sales by departments.

(*b*) Find the percentage of increase or decrease for each department.

(*c*) Find the percentage of increase or decrease in the total sales.

11. The following budget is that of an estimated operating statement:

Class	*Net Sales*
Class A	$2,000,000
Class B	300,000
Class C	150,000
Class D	40,000

Calculate the per cent of sales in each class as compared with total net sales.

12. A loss of $2\frac{1}{8}$% is taken on the sales of certain oil stocks which cost $12,000. What is the amount realized?

13. The manager of the June Department Store budgets the operating expenses of Department A at 19% over those of the previous year. The previous year's expenses were $6,041.18. What sum is budgeted for the current year?

14. An insolvent bank can pay 81% of its claims, with liquid assets totaling $61,000. What are the total claims?

15. A man pays an income tax of $188 on his taxable income. The income tax rate was 20%, and he took deductions of $500 for his child, $300 for other things, and $1,000 for himself and his wife. What is his gross income?

16. A speculator bought 1,000 bu of wheat at $2.36 per bushel. He was forced to sell 600 bu when the market was down to $2.12. The balance he sold at $2.45. What per cent of profit or loss did he make on his investment?

17. Don Fonda bought a house and lot for $12,000. The average expenses per year were: taxes, $320; insurance, $48; and repairs, $84. What rent per year will yield him an annual income of 6% on the original cost of the property?

18. Find the missing terms in the following table:

Item	*Gross Sales*	*Expenses*	*Commission*	*Rate of Commission*	*Net Proceeds*
(*a*)	$895.00	$14.50	$____	12%	$____
(*b*)	640.00	7.60	32.00	____	____
(*c*)	____	76.75	40.00	____	523.25
(*d*)	955.00	2.40	____	20%	____

19. At what price must a filing cabinet costing $60 be marked so that, after deducting 25% from the marked price, a gross profit of 10% on the cost may be realized?

20. The premium on a cottage insured for $5,000 is $62.50. Find the rate of premium.

21. How much insurance at 1.32% premium rate can be placed on a building for $63?

22. Find the cost for each of the following:

Item	*Selling Price*	*Per Cent of Selling Price*		*Expenses on Cost*	*Cost*
		Net Profit	*Net Loss*		
(*a*)	$ 56.80	12%		15%	____
(*b*)	44.95	10%		22%	____
(*c*)	125.00	$5\frac{1}{2}\%$		35%	____
(*d*)	67.75		$3\frac{1}{4}\%$	20%	____

23. After a television advertising campaign of 6 months a merchant found that he had increased his sales 3% each month over the previous month. If his sales were $120,000 for the month before the campaign started, how much were the sales for the last month?

24. A broker sells 1,000 bbl of vinegar at $18 per barrel with a commission of 2%. His expenses are $80.90 for freight, $29 for storage, $7 for insurance. He charges $2\frac{1}{2}$% for collection. What are the net proceeds?

25. A salesman sells $1\frac{1}{2}$ times as much in March as in February. He draws $225 on his commission during these months. After this is deducted from his earnings, the company gives him $75. If his commission is 10%, what was the amount of his sales in February?

26. A broker receives a consignment of 5,000 bu of corn. He pays freight charges of $100, a cartage and storage charge of $3\frac{1}{4}$¢ per bushel, and insurance charges of $12. He sells 1,000 bu at $2.04, 3,500 bu at $1.97, and the remainder at $2.15. He charges 3% commission and $1\frac{1}{2}$% for collection and guaranty. Find the net proceeds.

27. In the following table find the tax rate per $100:

Assessed Value	*Tax Paid*	*Rate per* $100
$161,000	$ 573.00	———
81,000	627.30	———
302,000	8,888.45	———

28. A municipality assesses property at 75% of its actual value. If the tax rate is $2.16 per $100 and the actual value of a man's property is $25,000, how much tax does he pay?

29. The assessed value of a community is $115,000,900, and $2,230,647 in taxes must be raised. What must be the tax rate per $100?

30. The manager of a chain store gets 10% of the profits after his salary is deducted as an expense. The year's profits before the salary is deducted are $30,000. What is his salary?

31. A suit cost $45. What price should be marked on the price tag so as to give a gross profit of 20% on the cost after allowing a discount of 16% to the customer?

32. For the fiscal year ending Sept. 30, an aircraft factory reported that its profits dropped 60% and totaled only $738,000. What were its profits for the previous year?

33. A merchant places an advertisement 3 columns wide and 6 in. deep in a metropolitan newspaper which charges $0.95 per agate line. As a result he sells 132 of the advertised articles at $3.85 a piece. Find the cost of the advertisement, the advertising cost per unit sold, and the advertising cost per item as a percentage of the selling price.

An advertising space 1 in. long and 1 column wide is said to consist of 14 agate lines.

34. A clothier places an advertisement for men's suits in a daily paper which charges $0.95 per agate line. The advertisement is 3 columns wide and 7 in. deep. Owing to the advertisement, he sells 150 of the suits at $50.

Find the advertising cost per unit sold and the advertising cost as a percentage of the selling price.

35. A novelty shop manager places an advertisement 4 columns wide and 4 in. deep in a daily city newspaper at a cost of $0.94 per agate line. The articles advertised are sold at $6 a piece. Twelve per cent of the selling price is allowable for advertising. Find the number of units of merchandise which must be sold if the advertising cost is to be kept within the 12% limit.

36. A department store puts an advertisement 6 columns wide and 10 in. deep in a local paper which charges $1.14 per agate line. The articles advertised are sold at 2 for $1.25. Find the number of articles which must be sold if the advertising cost is not to exceed 5% of the selling price.

Self-Test

Time: 50 Minutes

(Score 5 points for each problem correctly solved)

1. A college student council established a quota of 230 carnival tickets to be sold by the students of the Business Administration School. If the enrollment of this school is 350 and that of the entire college 1,400, on the basis of enrollment what should be the quota for the whole college?

2. It was estimated that the freshman class of the Business Administration School would sell 90 tickets. What per cent of this school's quota would be contributed by the freshman class? (Answer to nearest tenth per cent.)

3. If 45 of the 120 freshmen sold tickets, what per cent failed to do so? (Answer to nearest tenth per cent.)

4. A gross profit of 25% on cost is what per cent gross profit on selling price?

5. Higher prices forced Mr. Short to spend 90% of his take-home pay for living expenses. The balance of his take-home pay, $33 a month, was saved. What was his monthly salary, if he had deductions of taxes and insurance amounting to $20 taken from his monthly salary?

6. An article costing $25 was sold for $40. What was the per cent of profit on the selling price?

7. An article costing C cents was sold for D dollars. What was the per cent of gross profit on the selling price?

8. A suit cost $30. What price should be marked on the price tag so as to give a gross profit of 30% on the selling price after allowing a discount of 10% to the customer?

9. What per cent of $\frac{3}{16}$ is $\frac{2}{5}$ of $\frac{5}{8}$?

10. An article cost $20. If the markup is 40% of the cost and the net profit is 25% of the selling price, find the net profit.

11. Three successive discounts of 10%, 10%, and 10% were allowed on a bill amounting to \$200. Find the net amount.

12. What single discount is equivalent to two successive trade discounts of 10% and 5%?

13. $6\frac{3}{10}$ is 7% of what number?

14. Change $2.3\frac{1}{2}$% to a decimal.

15. Change 18% to a fraction.

16. Change $\frac{3}{20}$ to a per cent.

17. Change 2 to a per cent.

18. Change K per cent to a fraction.

19. Change 365 to a per cent.

20. What single discount is equivalent to three successive discounts of 10% each?

Chapter V

SIMPLE INTEREST AND DISCOUNT

54. Simple Interest. Interest is a charge involved when money is loaned. It is a payment made by the borrower for the use of money. The interest depends not only upon the amount of money borrowed but also upon the length of time for which the borrower has the use of the money, as well as upon the rate per cent agreed upon by both parties concerned. The sum of money loaned is called the *principal;* the per cent of the principal paid each unit of time is called the *rate;* and the number of periods of time, simply referred to as *time,* is usually expressed in years. The principal plus the interest is called the *amount.* In symbols, we write: I = interest, P = principal, t = time in years, r = rate, A = amount.

Suppose, now, a man borrows \$100 with the understanding that he is to repay the money at the end of 3 years, together with an interest charge of 4% per year. How much is the interest? What is the amount paid to the lender? Evidently, the interest charge for 1 year is \$100 × 0.04, or \$4; and for the 3 years the interest is 3 × \$4, or \$12. Hence, in one operation we can write

$$I = \$100 \times 0.04 \times 3 = \$12.$$

That is, the interest turns out to be the product of the three factors: *principal, rate,* and *time.* This fact may be expressed symbolically in the formula

$$I = Prt. \tag{1}$$

The amount paid to the lender at the expiration of the time for which the money was borrowed includes the principal and the interest and is, therefore, \$100 plus \$12, or \$112. Thus, the formula for amount is

$$A = P + I = P + Prt. \tag{2}$$

Formulas (1) and (2) express very concisely the relationships involved in simple interest problems and may be used as the basis for dealing with such problems.

Illustrations:

1. A man lends \$640 for $4\frac{1}{2}$ years at 4% simple interest. What interest does he receive on the loan? How much does the borrower repay at the end of the period of time?

Here $P = 640$, $r = 0.04$, and $t = 4\frac{1}{2}$. Using Formulas (1) and (2) we have

$$I = 640 \times 0.04 \times \tfrac{9}{2} = \$115.20.$$

$$A = 640 + 115.20 = \$755.20.$$

Since the formula for interest involves the product of the quantities concerned, cancellation may be used wherever possible to save labor in computation.

2. Mr. C. D. Stillwell made a loan of \$550 for a period of 3 years 4 months and received \$110 interest. What was the interest rate?

Here, $P = 550$, $I = \$110$, $t = 3\frac{1}{3}$, and we are to find r. Substituting into formula (1), we have

$$110 = 550 \times r \times \tfrac{10}{3},$$

or, upon solving for r,

$$r = 110 \times \tfrac{1}{550} \times \tfrac{3}{10} = \tfrac{3}{50} = 0.06 = 6\%.$$

3. In what time will \$4,800 invested at 6% yield \$1,656?

Here, $P = 4{,}800$, $r = 0.06$, and $I = 1{,}656$. We are to find t. Solving formula (1) for t before making substitutions, we have

$$t = \frac{I}{Pr}.$$

Substituting for the given quantities, we have

$$t = \frac{1{,}656}{4{,}800 \times 0.06} = 1{,}656 \div 288 = 5.75 \text{ years} = 5 \text{ years } 9 \text{ months}.$$

4. What sum of money will yield \$60 interest if invested at 2% simple interest for 3 years?

Here, $I = 60$, $r = 0.02$, and $t = 3$. We are to find P. Solving formula (1) for P, we have

$$P = \frac{I}{rt}.$$

Hence, $$P = \frac{60}{0.02 \times 3} = \frac{60}{0.06} = \$1{,}000.$$

Frequently an investor wishes to know the *average rate of interest* he is earning on several investments for a given year. For example, suppose he purchases at par 10 shares of 5% preferred stock par value \$100 and 20 shares of $6\frac{1}{2}$% stock par value \$10 and that he owns a $4\frac{1}{2}$% mortgage for \$2,000. His total investment (P) is seen to be \$3,200, and the total earnings (I) are calculated as \$50 + \$13 + \$90 = \$153. Using formula (1) with $t = 1$, and solving for r, we obtain

$$153 = 3{,}200r,$$

from which we find $r = 0.0478$, or 4.78% as the average rate of interest on all three investments for the year.

PROBLEMS

1. Complete the following table:

Principal	*Interest*	*Time*	*Rate*	*Amount*
(a) \$ 780	———	4 years 6 months	0.03	———
(b) \$ 500	\$ 78.75	———	0.035	———
(c) \$1,500	\$540.00	9 years	———	———
(d) ———	\$385.00	5 years	———	\$2,585
(e) \$3,600	———	———	0.02	\$4,248

2. How long will it take for \$1,200 to amount to \$1,308 at 4% simple interest?

3. C. D. Hare paid R. B. Cook \$908 on July 15, in return for \$800 previously borrowed from Cook. If the simple interest rate employed was $4\frac{1}{2}$%, when had Hare borrowed the \$800 from Cook?

4. How much money must Joe Blacow raise to repay a 5-month loan of \$325 bearing interest at $4\frac{3}{4}$%?

5. Gretchen borrows a five-dollar bill from Jan and gives him \$5.10 back in a month. What annual rate of interest is Jan getting?

6. A savings bond worth \$25 in 10 years cost \$18.75. What rate of simple interest does the bond earn?

7. Robert repaid a loan with \$1,535. He borrowed the money at $3\frac{1}{2}\%$ for 8 months. How much did he borrow?

8. William repaid a loan of \$250 with \$253.75. If the interest rate was 2%, how long did he have the loan?

9. At what rate of interest would \$1 amount to \$2 in 1 year?

10. How long would it take 50¢ to earn 50¢ simple interest at the rate of 1%?

11. How much would it cost a borrower to borrow \$100 at $3\frac{1}{2}\%$ per year for 1 month? How much would he pay back at the end of the month?

12. Find the interest charges on a loan of \$1,000 at $2\frac{1}{4}\%$ per year for 10 months.

13. An investor owns 50 shares of 5% preferred stock, par value \$100, at \$95, 100 shares of $5\frac{1}{2}\%$ preferred stock, par value \$100, at \$101, and a 5% mortgage for \$6,000. What is the average rate of interest earned by the investor?

14. Arthur Morrison bought 150 shares of $5\frac{1}{2}\%$ preferred stock at \$96, 100 shares of 6% preferred stock at \$102, and two \$1,000, $2\frac{1}{2}\%$ government bonds at par of \$1,000 each. Par value of the preferred stocks was \$100. What is the average rate of interest earned in a year?

55. Ordinary and Exact Interest. In the preceding illustrations and problems the time was expressed in terms of years. If this is not the case and the time is given in days, there are two ways of converting the time into years. The one, known as *ordinary* interest, is based on 360 days in a year, and the other, known as *exact* interest, is based on 365 days in a year. Any given period of time in days, therefore, can be expressed in terms of years by dividing by either 360 or 365, depending upon the kind of interest called for. That is, if d is the number of days, we may write

$$\text{Ordinary simple interest } I = \frac{Prd}{360},$$

$$\text{Exact simple interest } \quad I = \frac{Prd}{365}.$$

Illustration: Find the ordinary and exact interest on $300 for 60 days at $3\frac{1}{2}\%$.

Using the formula involving 360 days, we find

$$I = 300 \times 0.035 \times \tfrac{60}{360} = \$1.75 \qquad \text{(ordinary interest).}$$

Using the formula involving 365 days, we find

$$I = 300 \times 0.035 \times \tfrac{60}{365} = \$1.73 \qquad \text{(exact interest).}$$

The reason why ordinary interest is greater than exact interest is that, in the case of ordinary interest, the divisor is 360, whereas, in the case of exact interest, the divisor is the larger number 365. Obviously, the difference in interest computed by the two methods may be quite significant where large sums are involved.

56. Approximate and Exact Time. When loans are made for short periods of time between two dates, it is necessary to count the days between the two dates. This may be done by counting the *exact* number of days for which the loan is made or by calculating the *approximate* number of days on the assumption that each month has 30 days. A table from which the number of days between two dates can readily be found is a convenient aid in finding exact time. The following table will be found useful for this purpose.

NUMBER OF DAYS BETWEEN CORRESPONDING DAYS IN DIFFERENT MONTHS

	Jan.	*Feb.*	*Mar.*	*Apr.*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec*
Jan.	365	31	59	90	120	151	181	212	243	273	304	334
Feb.	334	365	28	59	89	120	150	181	212	242	273	303
Mar.	306	337	365	31	61	92	122	153	184	214	245	275
Apr.	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
Aug.	153	184	212	243	273	304	334	365	31	61	92	122
Sept.	122	153	181	212	242	273	303	334	365	30	61	91
Oct.	92	123	151	182	212	243	273	304	335	365	31	61
Nov.	61	92	120	151	181	212	242	273	304	334	365	30
Dec.	31	62	90	121	151	182	212	243	274	304	335	365

The numbers in the above table give the number of days from any given day in one month to the same day in any other month.

Thus, the number of days from Jan. 10 to Aug. 10 is seen to be 212. From Aug. 10 to Jan. 10 of the following year, the number of days is seen to be 153. (Always start in the column at the left and read across to the column under the final month.) The table may be used for any period of time which does not exceed one year. For leap years, the number of days is one greater than the tabular number when the interval of time includes Feb. 29.

Illustrations:

1. Find the exact time between June 11, 1961, and Apr. 15, 1962.

In the table we read 304 as the number of days from June 11 to Apr. 11. To this we add 4, the number of days from Apr. 11 to Apr. 15, for a total of 308 days.

If we assume that each month contains 30 days, the *approximate* time from one date to another can be found by subtraction. In this process we must write the two dates in columns (year, month, and day) and subtract column by column, borrowing wherever necessary.

2. Find the approximate time between June 11, 1961, and Apr. 15, 1962.

Write the dates as follows:

	Year	*Month*	*Day*
	1962	4	15
	1961	6	11
Subtract, and find		10 months	4 days, or 304 days.

Note that since 6 cannot be taken from 4 in the month column, 12 months are borrowed from the year.

The exact number of days between two dates is ordinarily greater than the approximate number of days based on a 30-day month. Also, ordinary simple interest is greater than exact simple interest. It follows, therefore, that the most favorable arrangement for the creditor is ordinary simple interest with exact time, since it results in the largest amount of interest. This method is the one commonly used by commercial banks. Ordinary simple interest with approximate time is commonly used in determining interest on bonds and in calculating interest on individual loans. Exact simple interest with exact time is the method used in many governmental activities. It is very seldom that one encounters the use of exact simple interest with approximate time. *If no mention is made of the type of simple interest involved, we shall hereafter assume it to be ordinary simple interest for the exact number of days.*

PROBLEMS

1. Find the exact and approximate time between the following dates: (*a*) Aug. 7, 1962, and Feb. 17, 1963; (*b*) July 23, 1961, and Sept. 6, 1963; (*c*) Mar. 23, 1962, and Oct. 9, 1962; (*d*) Dec. 14, 1963, and Mar. 5, 1964; (*e*) May 26, 1963, and June 26, 1964; (*f*) June 5, 1961, and July 23, 1963; (*g*) Nov. 27, 1963, and Nov. 28, 1964.

2. Find the ordinary and exact interest for each of the following: (*a*) \$450 at 4% for 29 days, (*b*) \$2,000 at 3% for 123 days, (*c*) \$275.60 at 3% for 36 days, (*d*) \$5,480 at $4\frac{1}{2}$% for 90 days, (*e*) \$632.50 at $3\frac{1}{2}$% for 120 days, (*f*) \$56,250 at $3\frac{1}{2}$% for 12 days, (*g*) \$2,500 at 5% for 167 days.

3. Find the ordinary interest for the exact time on \$5,000 at 3% from May 6 to Dec. 9, 1963.

4. Find the exact interest for the exact time on \$400 at 5% from Mar. 23 to Sept. 26, 1963.

5. Find the ordinary interest for the approximate time on \$10,000 at 3% from Jan. 5 to July 20, 1963.

6. Find the exact interest for the approximate time on \$600 at 4% from Feb. 9 to Mar. 25, 1964.

7. Jones borrows \$1,000 from Smith on Mar. 3 at 3% and repays the loan on May 3. Find the ordinary simple interest for the exact time and for the approximate time. Which is greater? How much?

8. In Problem 7, find the exact simple interest for the approximate time and for the exact time. Find the difference, and compare it with the difference obtained in Problem 7.

9. On Mar. 15, Brown paid \$300 income tax. On Jan. 20 following, the government notified him that his return had been \$30 too low and that he must pay it plus 4% interest. What type of interest should he pay to his greatest advantage? What interest would he pay under this type?

10. Find the ordinary simple interest on \$1,000,000 for the exact and approximate number of days from June 17, 1962, to Dec. 31, 1962, at 3%. Find the difference in the two amounts.

11. Find the difference between the exact and ordinary simple interest charge on \$1,000,000 for 60 days at 4%.

12. White overpaid his income tax for the year in the amount of \$150. Assuming that he overpaid it as of Mar. 15 and that it was returned to him on Jan. 18 following, at $2\frac{1}{2}$% ordinary simple interest for the approximate number of days, what amount did he receive?

13. On Oct. 22, Mr. Clark borrows \$20,000 and gives a mortgage on his business. The terms of the loan call for 5% interest to be paid on the 15th day of each December and each June. What is the first interest payment?

14. Mr. Fields borrows \$3,000 at $5\frac{1}{2}$% and gives a mortgage on his farm. The first payment of interest is due in 122 days, and payments thereafter are to be made every 6 months. What is the first payment? The second payment?

57. Present Value. We have seen that when a sum of money is placed at simple interest at a given rate for a given time, the principal, or sum invested, "grows" to a certain amount. Quite frequently we wish to know the sum of money that must be placed at interest to obtain a specified amount at some future date. That is, we wish to know the *present worth*, or *present value*, of an amount. The problem, therefore, is to find the value of P in the formula for amount. We have

$$A = P + I,$$

or $$A = P + Prt,$$

Factoring, $$A = P(1 + rt).$$

Transposing members, $$P(1 + rt) = A.$$

Dividing by the coefficient of P,

$$P = \frac{A}{1 + rt} \tag{3}$$

Formula (3) will give the present value P, or principal, that at a rate r per year and a time t in years will, at simple interest, amount to A.

Illustration: What principal will amount to \$682 in 4 years at 3% simple interest?

We observe that we are given all the quantities in formula (3) except P. Substituting $A = 682$, $t = 4$, and $r = 0.03$ in this formula, we find

$$P = \frac{682}{1 + (0.03 \times 4)} = \frac{682}{1.12} = \$608.93 \text{ principal.}$$

We may check this result by noting that \$608.93 in 4 years at 3% amounts to

$$608.93 + (608.93 \times 0.03 \times 4) = 608.93 + 73.07 = \$682.$$

PROBLEMS

1. Find the present value of:

   (*a*) \$376 due in 3 years 6 months at 5%,
   (*b*) \$100 due in 5 years at 2%,
   (*c*) \$85 due in 1 year at $3\frac{1}{2}$%,
   (*d*) \$60 due in 2 years 6 months at $4\frac{1}{2}$%,
   (*e*) \$126 due in 9 months at $1\frac{1}{4}$%,
   (*f*) \$150 due in 8 months at $2\frac{1}{2}$%,
   (*g*) \$230 due in 9 months at 4%,
   (*h*) \$1,500 due in 6 months at $3\frac{1}{2}$%,
   (*i*) \$915 due in 4 months at 5%,
   (*j*) \$759 due in 10 months at $6\frac{1}{2}$%.

2. What amount must Simpson invest now at simple interest of 3% to have \$617.10 eight months from now?

3. Williams needs to have \$750 available at the end of 9 months. If he can get 2% simple interest on his money, what amount should he invest now?

4. Green can sell his house now for \$12,500 or for \$12,550 one month from now. If the simple interest rate is 2%, which is the better offer for him on the basis of present value?

5. Dunlap can sell his car now for \$900 or for \$905 two months from now. If the simple interest rate is 3%, which is his better deal from the standpoint of present worth? How much better?

6. Gordon is to receive a trust fund of \$15,000 in 8 months. What amount can he borrow now at $3\frac{1}{2}$% if he uses the trust fund to pay off the debt at the end of 8 months?

7. What present amount will be worth \$190 six months from now if savings earn $3\frac{1}{2}\%$ simple interest?

8. What would \$800 on Aug. 16 have been worth on Mar. 2 immediately prior to that date if the ordinary simple interest rate is 2%? (Use approximate number of days.)

9. What would \$50,000 on July 7, 1962, be worth on Sept. 18, 1961, if the exact simple interest rate is 3%? (Use exact number of days.)

10. The present worth of \$1,000 due at a future date is \$995.57. If the ordinary simple interest rate is 4%, find the time until maturity. (Compute answer to nearest day.)

58. True Discount. The difference between the amount due on a sum placed at simple interest and the principal itself is called *true discount.* Thus, the true discount on an amount A is the same as the interest on a principal P. To obtain the true discount is to obtain the simple interest. Hence, $D = A - P$.

When money earns interest at a rate r, we say that money is worth $r\%$. Thus, the expression. "Money is worth 2%," means that the simple interest rate is 2%.

Illustrations:

1. What are the present value and the true discount on a note for \$100 due in 1 year if money is worth 5%?

The problem here could be rephrased to read, "What sum of money placed at 5% interest *now* would amount to \$100 in 1 year, and what would be the interest on this sum?" Substituting the known values, $A = 100$, $t = 1$, and $r = 0.05$ in formula (3), we have

$$P = \frac{A}{1 + rt} = \frac{100}{1 + (0.05)(1)} = \frac{100}{1.05} = \$95.24.$$

The present value of the note is, therefore, \$95.24. This would be the "cash" value of a note for \$100 due in 1 year if money is worth 5%. The true discount, or the difference between the maturity value and the present, or "cash," value, therefore, is easily obtained. We have

$$D = A - P = 100 - 95.24 = \$4.76.$$

Notice that *true discount is, in reality, the difference between the maturity value of the note and its present value.*

2. On Jan. 7, 1955, Mr. C. R. Adams borrows \$100 from Mr. C. C. Jones with the understanding that the money is to be repaid with simple interest at 5% at the end of 6 months. Jones made out the following note, which was signed by Adams:

Cleveland, Ohio, Jan. 7, 1955

Six months after date I promise to pay to the order of C. C. Jones \$100 with interest at the rate of 5% per annum after date.

Signed: C. R. Adams

Mr. Jones has the note discounted at a bank 4 months before it is due at 4% true discount. Find the true discount.

First we compute the future value of the note, or the amount of \$100 at ordinary simple interest for ½ year at 5%. We find

$$A = 100[1 + (0.05)(\tfrac{1}{2})] = 100 + 2.50 = \$102.50.$$

Next, we compute the present value of \$102.50 as of 4 months before the due date, using the rate of 4%. We find

$$P = \frac{102.50}{1 + (0.04)(\frac{1}{3})} = \frac{102.50}{\frac{76}{75}} = \frac{102.50 \times 75}{76} = \$101.15.$$

Since true discount D equals $A - P$, we find

$$D = 102.50 - 101.15 = \$1.35 \text{ true discount.}$$

The agreement signed by Mr. Adams is an example of a *promissory note.* The *face* is the sum of money mentioned in the note (\$100). The *maturity value* is the amount of money (\$102.50) that the debtor is to pay on the *maturity date* (July 7, 1955) and usually includes interest at an agreed rate (5%). If the note is non-interest-bearing, the face value is also the maturity value.

PROBLEMS

1. Find the true discount on each of the following notes:

(*a*) \$500 due in 3 years, money worth 2%,
(*b*) \$1,000 due in 60 days, money worth 3%,
(*c*) \$250 due in 4 years 3 months, money worth 5%,
(*d*) \$640 due in 2 years 4 months, money worth 4½%,

(*e*) \$12,000 due in 24 days, ordinary simple interest rate 4%,
(*f*) \$150 due in 9 months, money worth 2%,
(*g*) \$250 due in 6 months, money worth $3\frac{1}{4}\%$,
(*h*) \$10,000 due in 1 year, money worth $5\frac{3}{4}\%$,
(*i*) \$530 due in 35 days, exact simple interest rate 3%,
(*j*) \$650 due in 1 year 2 months, money worth 5%.

2. Find the true discount on each of the following interest-bearing notes:

Note	*Face Value*	*Rate of Interest*	*Term of Note*	*Term of Discount*	*Discount Rate*
(*a*)	\$ 1,000	3%	90 days	30 days	2%
(*b*)	500	2%	120 days	90 days	$1\frac{1}{2}\%$
(*c*)	650	$2\frac{1}{2}\%$	6 months	5 months	2%
(*d*)	430	4%	5 months	5 months	$3\frac{1}{2}\%$
(*e*)	150	3%	9 months	8 months	2%
(*f*)	10,000	2%	60 days	30 days	$1\frac{1}{2}\%$

3. A clothier pays for a consignment of jackets with a non-interest-bearing 90-day note for \$1,500. This note is discounted 60 days before it is due at a discount rate of 3%. Find the true discount.

4. Smith, a merchant, gives a 120-day note for \$2,000 bearing interest at 2% for a consignment of groceries. If this note is discounted the same day at a $1\frac{1}{2}\%$ discount rate, find the true discount.

59. Bank Discount. *Bank discount is simple interest deducted in advance and is computed on the maturity value of the note.* If, for example, \$100 is borrowed at a bank for 1 year at 5% interest, the \$5 charge is deducted at the time the loan is made, and the borrower receives \$95. The understanding is that he is to pay the bank \$100 at the end of 1 year. If the student will compare this procedure with the first true discount illustration of the last section, he will see that the bank discount (\$5) is greater than the true discount (\$4.76). In the former case, the interest is calculated on the basis of the present value of the loan. In the latter case, the interest is computed on the basis of the maturity value of the note. The methods of computing bank discount at a given rate for a given number of days are the same as those for computing simple interest. *Observe that the bank charges discount for the total number of days that the bank is without the maturity value of the note.*

60. Proceeds. *The difference between the amount due at maturity and the bank discount is called the proceeds of a note.* By proceeds we mean the amount received when a note is discounted ("cashed") at a bank. We must remember that the bank computes the discount on the basis of the maturity value of the note, so that if a note is interest-bearing, its *maturity date* value must be used in computing the discount. If we denote bank discount by D, the maturity value by A, the period of discount (in years) by t, and the annual discount rate by d, we may write as a formula for the proceeds

$$P = A - D,$$

or

$$P = A - Adt = A(1 - dt). \tag{4}$$

Illustration 1: A 90-day note for $300 bearing interest at 3% was given to the Jackson Product Company on a certain day. After holding the note for 30 days, the company took the note to a bank and had it discounted. If the bank charged a $3\frac{1}{2}\%$ discount rate, how much was the discount, and how much did the company receive for the note?

In solving this problem, let us list a few questions and their answers as an aid in the understanding of the simple principles involved in bank discount.

(1) How long does the bank hold the note? *Answer:* The company held it for 30 days. The *term of the note is* 90 days. Therefore, the bank holds the note 60 days.

(2) For how many days does the bank charge in discounting the note? *Answer:* Since the bank is without its money for 60 days, the charge is made for 60 days.

(3) On what sum of money is the discount charge based? *Answer:* Since the note bears interest at 3%, the maturity value of the note is the amount of $300 for 90 days at 3%, which equals $300 \times \frac{90}{360} \times 0.03$, or $2.25, plus $300, or $302.25. *Note that the discount is based on the* $302.25, *not on the* $300.

(4) How much discount does the bank charge? *Answer:* The bank charges $3\frac{1}{2}\%$ on $302.25 for 60 days. Hence, the discount equals $\$302.25 \times \frac{60}{360} \times 0.035 = \$1.763$, or $1.76.

(5) How much does the company receive for the note? *Answer:* The proceeds is the difference between the maturity value of the note and the bank discount. Hence, the company receives \$302.25 less \$1.763, or \$300.487, or \$300.49.

Illustration 2: Find the bank discount and proceeds for a 60-day note dated Mar. 5, with face value \$100 bearing interest at the rate of 3% if the note is discounted by a bank on Apr. 3 at a discount rate of 4%.

First, we calculate the maturity value of the note:

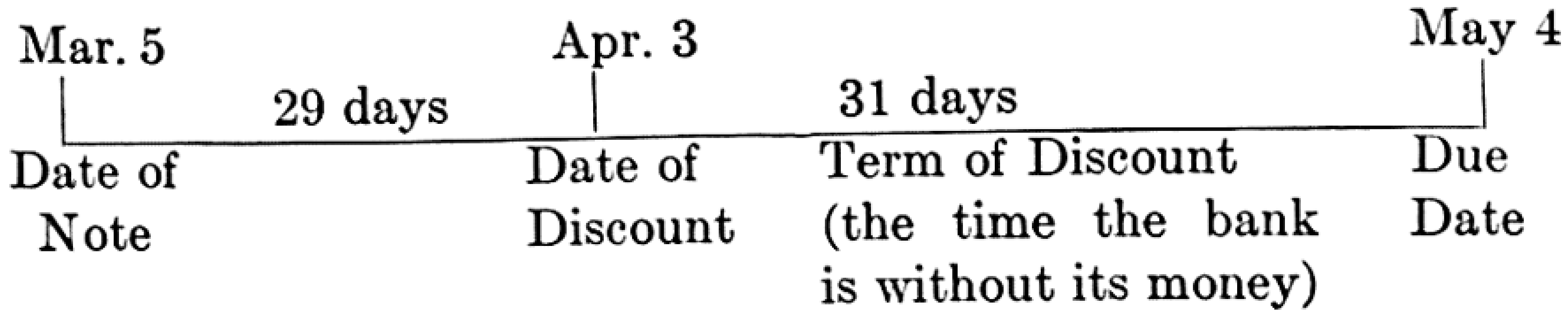
$$\begin{aligned} A &= P(1 + rt), \\ &= 100[1 + (0.03)(\tfrac{60}{360})], \\ &= 100(1 + \tfrac{1}{200}) = 100 + \tfrac{1}{2} = \$100.50. \end{aligned}$$

Second, we find the number of days for which the note is discounted by determining the exact number of days between Mar. 5 and Apr. 3 and, by subtracting this difference from 60, the term of the note.

From Mar. 5 to Apr. 3 is 29 days.

$$60 - 29 = 31 \text{ days, term of discount.}$$

In other words, the bank is without its money 31 days if it discounts the note on Apr. 3. Graphically, we have:



Third, we find bank discount and proceeds.

$$\begin{aligned} D &= (100.50)(0.04)(\tfrac{31}{360}) = \$0.35, \\ P &= 100.50 - 0.35 = \$100.15. \end{aligned}$$

PROBLEMS

Find the bank discount and proceeds for each of the following notes:

Face Value	*Term of Note*	*Interest Rate*	*Term of Discount*	*Discount Rate*
1. $ 300	90 days	3%	30 days	4%
2. 1,000	60 days	4%	40 days	4%
3. 950	120 days	4%	60 days	4½%
4. 15,000	45 days	5%	30 days	5%
5. 150	90 days	3½%	80 days	3%
6. 600	60 days	2½%	60 days	3%
7. 50,000	120 days	2%	90 days	2½%
8. 5,000	30 days	3%	30 days	3%
9. 5,000	90 days	3%	90 days	3%

Face Value	*Date of Note*	*Term of Note*	*Interest Rate*	*Date of Discount*	*Discount Rate*
10. $ 400	Feb. 6	60 days	6%	Mar. 2	7%
11. 2,600	Aug. 12	90 days	4%	Aug. 22	5%
12. 5,000	July 23	120 days	7½%	Sept. 15	6%
13. 680	Mar. 19	2 months	5½%	Apr. 6	8%
14. 380	Jan. 20	4 months	6%	Mar. 17	6½%
15. 150	June 20	60 days	3%	June 20	3%
16. 100	Aug. 3	120 days	3½%	Aug. 23	3%
17. 1,000	Sept. 21	90 days	2%	Oct. 21	3%
18. 500	Jan. 7	90 days	2½%	Jan. 17	3%

19. A 90-day note for $10,000 bearing interest at 3% is discounted on the same day it is executed. What is the discount rate if the proceeds equal the face of the note?

20. A 120-day note for $500 bearing interest at 4% is discounted 30 days before it is due, yielding proceeds of $504.56. Find the discount rate.

21. A 120-day note for $600 bearing interest at 3% is discounted at 3%, and the proceeds amount to $599.94. How long before it was due was the note discounted?

22. A 9-month note for $350 bearing interest at 4% is discounted 6 months before it is due, and the proceeds are $353.29. Find the discount rate.

23. A 90-day note for $800 bearing interest at 5% is discounted at a bank 60 days before it is due at a discount rate of 5½%. The bank rediscounts it the same day at the Federal Reserve Bank,

where the discount rate is 3%. Find the profit of the member bank. (*Note:* The fact that the note has already been discounted has no effect on the computation of proceeds by the Federal Reserve Bank.)

If a note is non-interest-bearing, the value of the note at maturity is the same as the face value of the note. The bank discount, in this case, is computed on the basis of the face value for the time of the discount period.

Illustration 3: A non-interest-bearing note for \$200 was discounted at 5%, twenty days before maturity. Find the proceeds.

Here, $A = 200$, $d = 0.05$, and $t = \frac{20}{360}$. Hence, $D = 200 \times 0.05 \times \frac{20}{360} = \$0.56$. The proceeds P, therefore, equals $200 - 0.56 = \$199.44$.

PROBLEMS

Find the bank discount and proceeds for each of the following non-interest-bearing notes:

Face Value	Date of Note	Term of Note	Date of Discount	Discount Rate
1. \$ 200	Aug. 1	60 days	Aug. 16	7%
2. 1,000	Mar. 15	90 days	Apr. 14	6%
3. 450	June 3	120 days	June 8	$5\frac{1}{2}$%
4. 185	Jan. 23	30 days	Feb. 4	$7\frac{1}{2}$%

5. An automobile dealer discounts a 90-day non-interest-bearing note for \$500, thirty days before it is due, at his bank which charges 2% bank discount. The bank rediscounts the note the same day at the Federal Reserve Bank where the discount rate is 1%. How much did the bank make on the transaction?

6. A 9-month non-interest-bearing note for \$5,000 is discounted by the Third National Bank 6 months before it is due at $1\frac{1}{2}$% bank discount. The bank immediately rediscounts the note at the Federal Reserve Bank, which charges 1% rediscount. How much profit does the Third National Bank make?

61. Interest Rate Corresponding to Discount Rate. In connection with bank discount questions are often asked concerning the rate of interest earned by the bank in performing the transaction. In Illustration 1 of the preceding section we saw that, in discounting a note for \$300 bearing interest at 3% sixty days before the maturity date, the proceeds were \$300.487 when the bank discount rate was $3\frac{1}{2}\%$. This means that the bank virtually made a loan of \$300.487 for the 60 days and collected the maturity value of the note (\$302.25) at the end of that time. Hence, to determine the interest rate corresponding to the given discount rate, we must find that rate at which \$300.487 will amount to \$302.25 in 60 days. Our task, then, is to find r in the formula, $A = P + Prt$, where $A = \$302.25$, $P = \$300.487$, and $t = \frac{60}{360}$. Substituting these quantities in the formula, we have

$$302.25 = 300.487 + 300.487 \times \tfrac{60}{360} \times r.$$

Solving for r, we have

$$1.763 = \frac{300.487 \times r}{6},$$

$$10.578 = 300.487r,$$

or

$$r = 10.578 \div 300.487 = 0.0352 = 3.52\%.$$

This means that although the bank discount rate is $3\frac{1}{2}\%$, the bank actually earns more than $3\frac{1}{2}\%$ simple interest on the transaction.

Can we now arrive at a formula which will give the relationship between bank discount and interest rate? We can obtain such a formula by carrying through a development similar to our example above in terms of symbols. We begin with the formula for proceeds. We have

$$P = A - Adt, \qquad \text{or} \qquad P = A(1 - dt),$$

from which $A = \dfrac{P}{1 - dt}$.

If the proceeds P amounts to the maturity value A in time t, the interest earned on P in that time is $A - P$. That is, the interest earned is

$$\frac{P}{1 - dt} - P, \qquad \text{or} \qquad \frac{P - P(1 - dt)}{1 - dt}, \qquad \text{or} \qquad \frac{Pdt}{1 - dt}.$$

Now, since t is expressed in years, we must divide by t to find the interest for 1 year. Hence, the interest for 1 year is

$$\frac{Pdt}{1 - dt} \div t, \qquad \text{or} \qquad \frac{Pd}{1 - dt}.$$

The interest on \$1 is, therefore,

$$\frac{Pd}{1 - dt} \div P, \qquad \text{or} \qquad \frac{d}{1 - dt}.$$

But the interest on \$1 for 1 year is the annual interest rate. Hence, the interest rate r corresponding to a discount rate d is given by the formula

$$r = \frac{d}{1 - dt}. \tag{5}$$

Using formula (5) to find the interest rate for our example, we substitute $d = 0.035$ and $t = \frac{60}{360}$ and find

$$r = \frac{0.035}{1 - 0.035 \times \frac{1}{6}} = \frac{0.035}{1 - 0.005833} = \frac{0.035}{0.994167} = 0.0352 = 3.52\%$$

as before.

By solving formula (5) for d in terms of r and t, we may readily find a formula for discount rate corresponding to a given interest rate. We have

$$r = \frac{d}{1 - dt},$$

Cross multiply, $r(1 - dt) = d,$

Remove parentheses, $r - rdt = d,$

Transpose, $d + rdt = r,$

Factor, $d(1 + rt) = r,$

Divide by coefficient of d,

$$d = \frac{r}{1 + rt}. \tag{6}$$

Formula (6) may be used to find the discount rate corresponding to a known interest rate for a given period of time, where t is the term of the discount in years.

Illustration: A bank wishes to earn 6% interest in discounting a note for 3 months. What should be the discount rate?

Here, $r = 0.06$, $t = \frac{3}{12}$, and we desire to find d. Substituting in formula (6), we have

$$d = \frac{0.06}{1 + 0.06 \times \frac{1}{4}} = \frac{.06}{1 + 0.015} = \frac{.06}{1.015} = 0.0591 = 5.91\%.$$

PROBLEMS

1. A Fairview bank discounted a note for $2,000 thirty-six days before it was due. If the bank earned 5% interest on the transaction, what was the discount rate?

2. What interest rate is equivalent to a 7% discount rate if the term of discount is (*a*) 9 months, (*b*) 1 year?

3. What discount rate is equivalent to a 3% simple interest rate if the term of discount is (*a*) 2 months, (*b*) 4 months?

4. A merchant takes a non-interest-bearing note for $5,000, due in 90 days, to his bank, which discounts it at $6\frac{1}{2}$% bank discount rate on the day the note was executed. What rate of simple interest does the bank earn?

5. Discount $850 due in 6 months, (*a*) at 4% bank discount, (*b*) at 4% true discount.

6. What is the amount of a note at the end of 15 months if its present value is $450, (*a*) at 2% simple interest, (*b*) at 2% bank discount?

7. Mr. Roe may have his 90-day non-interest-bearing note for $2,000 discounted at either 4% bank discount or 5% true discount. Which is the more favorable procedure for him? How much more favorable?

8. If Mr. Roe has a choice of $3\frac{1}{2}\%$ bank discount or 4% true discount when borrowing $500 for 90 days, which is his better choice and by what amount?

9. Mr. Allen needs $4,000 to meet an obligation. How much must he borrow from a bank so that the proceeds of a 90-day note discounted at 5% will be the required amount to meet his obligation?

10. The invoice on a shipment of golf clubs to a dealer was for $860, dated Apr. 4, terms 2/10, $n/30$. He decided to take advantage of the discount and to pay the invoice on Apr. 14 by borrowing just the amount needed, giving a note due in 20 days at 6% bank discount. How much did he save?

62. Interest Rate in Installment Buying. Much retail selling is done on the *installment plan.* In this plan, the customer makes a down payment for goods purchased and agrees to pay stated amounts at intervals (usually regular) for a length of time to cover the balance of the purchase price. Under this plan, it is customary for the retail establishment to charge the customer an amount of money for the privilege. This charge is called the *carrying charge.*

It may be of interest to a customer to determine the actual rate of interest that is paid under the installment plan. For example, suppose that a refrigerator that sells for $250 cash is bought for $50 down and $20 a month for 11 months. Does the difference between $250 and $270 truly represent the cost to the customer resulting from the payment plan? If not, what is the rate of interest actually being charged? To answer this latter question, we need to concern ourselves with annuities, a subject to be taken up in a later chapter. However, we shall give here a formula for computing the approximate rate of interest that the customer actually pays. This simple formula* may be written as follows:

$$r = \frac{6M}{3(n+1) + M(n-1)}, \tag{7}$$

in which r = interest rate per period,
n = number of equal periodic payments,

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* Due to H. E. Stelson, *The American Mathematical Monthly,* Vol. 56, pp. 257–261.

and
$$M = \frac{Rn - P}{P},$$

in which R = period payment,

P = unpaid balance after initial down payment,

and n has the same meaning as above. It may be observed that

$$M = \frac{\text{carrying charge}}{\text{unpaid balance}},$$

and that this ratio can readily be computed and substituted into the formula for r along with the value of n to calculate the approximate interest rate per period. The *yearly interest rate,* which is usually more meaningful, may be found by multiplying r by the number of payments in a year.

It can be shown that the error in r computed by the approximation formula is less than

$$\frac{2(n - 1)(n + 2)M^3}{9(n + 1)^3}.$$

This error is usually very small. For example, if $M = 0.1$ and n is around 12, the error is less than 0.00002. Furthermore, the "upper bound" of error becomes less as M decreases for a fixed value of n and also becomes less as n increases for fixed values of M. Because of these properties, the formula for r may be considered quite useful for making a rapid calculation of the actual interest rate being charged under a given installment plan.

Illustration 1: A refrigerator sells for \$250 cash or \$50 down and \$20 at the end of each month for 11 months. Find the interest rate charged under the installment plan.

The quantities to be substituted into formula (7) may be listed as follows:

n = 11, number of equal periodic payments,

P = \$200, unpaid balance after the down payment,

R = \$20, periodic payment,

$$M = \frac{Rn - P}{P} = \frac{(20)(11) - 200}{200} = \frac{220 - 200}{200} = \frac{20}{200} = 0.1.$$

Hence,
$$r = \frac{(6)(0.1)}{(3)(11+1)+(0.1)(11-1)} = \frac{0.6}{36+1} = \frac{0.6}{37} = 0.01622.$$

Since there are 12 periods in a year, the approximate yearly interest rate is, therefore, $12 \times 0.01622 = 0.195$, or 19.5%.

Sometimes it is convenient to have the periodic payment fixed at some convenient amount, as for example $20 per month, with the last payment equaling whatever amount remains after the other equal payments have been made. In this situation the approximate periodic rate of interest, r, under the installment plan may be obtained from the formula

$$r = \frac{2I(P+I)}{P(n+1)(P+I+L)}, \tag{8}$$

in which r = interest rate per period,
n = number of equal period payments,
R = amount of each equal periodic payment,
P = unpaid balance after initial down payment,
L = amount of last payment,
$I = Rn + L - P$ = carrying charge.

As in the case where all payments are equal, the yearly rate of interest may be obtained from r by multiplying by the number periods in a year.

Illustration 2: A man bought a watch priced at $60 cash for $20 down and $5 a month for 8 months, and a payment of $4 the ninth month. Find the rate of interest charged under the installment plan.

The quantities to be substituted into formula (8) may be listed as follows:

$n = 8$,

$R = \$5$,

$P = \$40$,

$L = \$4$,

$I = (\$5)(8) + \$4 - \$40 = \$4.$

Making the substitutions and solving for r, we find

$$r = \frac{2 \times 4(40 + 4)}{40 \times 9(40 + 4 + 4)} = \frac{8 \times 44}{40 \times 9 \times 48} = \frac{11}{540} = 0.02037.$$

Since there are 12 periods in a year, the approximate interest rate per year is, therefore, $12 \times 0.02037 = 0.244$, or 24.4%.

PROBLEMS

Find the yearly interest rate being charged in each of the following installment plans:

Plan	*Cash Price of Article*	*Down Payment*	*Number of Payments*	*Amount of Each Payment*	*Payment Due (Period)*
1.	$ 60	$ 10	10	$ 5.50	Monthly
2.	100	10	12	8.00	Monthly
3.	40	None	22	2.00	Weekly
4.	50	2	52	1.00	Weekly
5.	500	150	24	16.25	Monthly
6.	2,000	500	16	100.00	Monthly
7.	1,000	None	20	58.70	Monthly

8. Show that if the ratio of carrying charge to unpaid balance is small enough to be neglected (that is, if M is small compared with 3), then an approximation to the periodic interest rate r in formula (7) is given by the formula

$$r = \frac{2M}{n + 1}.$$

9. Use the formula of Problem 8 to compute the periodic interest rate for an installment plan purchase of a $200 camera bought for $20 down and $16 a month for 12 months. Compare this answer with that obtained by using formula (7).

10. A certain small loan company will make a loan of $500 to be repaid in 12 monthly payments of $48 each. Find the rate of interest charged.

11. A portable radio sells for $39 cash or $9 down and $5 a month for 6 months, and a payment of $2 for the seventh or last payment. Find the rate of interest paid by a customer using the installment plan.

63. Partial Payments. When an interest-bearing note is paid at maturity, the amount of the settlement equals, of course, the sum of the face of the debt plus the interest. But what happens when the debtor makes equal or unequal payments on the principal before the date agreed upon? The payments should draw interest in favor of the debtor, for the money which he might have used for the remainder of the period is now in the hands of the creditor sooner than agreed upon.

There are two methods of applying partial payments to the reduction of a debt. The method adopted by the United States Supreme Court, and which is now a law in nearly every state, is called the "United States Rule"; the other method, which is more or less prevalent among businessmen, is known as the "Merchants' Rule."

According to the United States Rule, when a payment is made on an interest-bearing debt, the payment must first be applied to the accumulated interest and what remains goes toward the reduction of the principal. In case the payment is smaller than the interest due, the previous principal continues to draw interest until this payment and succeeding payments exceed the accumulated interest.

According to the Merchants' Rule, the principal draws interest from the date of indebtedness to the date of final settlement, and when a payment is made, interest is allowed on the payment from the date of payment to the date of final settlement. When the sum of all the payments together with the interest on these payments is deducted from the total amount of the debt including interest, the remainder is the amount due at the date of settlement.

In order that the two methods may be compared, the following illustration will be carried out by both methods.

Illustration: On Mar. 15, Mr. Adams gave a note for $1,000 in order to purchase some building materials. The note carried interest at 3%, and three payments were made as follows:

Mar. 30	$200
June 28	6
Aug. 3	525

How much was due Sept. 11?

SOLUTION BY UNITED STATES RULE

Face of note		$1,000.00
Interest from Mar. 15 to Mar. 30—$1,000 for 15 days		1.25
Amount due Mar. 30		$1,001.25
Deduct Payment		200.00
Balance due Mar. 30		$ 801.25
Interest from Mar. 30 to June 28—$801.25 for 90 days		(6.01)
Since the interest exceeds the second payment, the second payment is held over and the principal remains unchanged.		
Interest from Mar. 30 to Aug. 3—$801.25 for 126 days		8.41
Amount due Aug. 3:		$ 809.66
Deduct sum of payments:		
June 28	$ 6.00	
Aug. 3	525.00	531.00
Balance due Aug. 3		$ 278.66
Interest from Aug. 3 to Sept. 11—$278.66 for 39 days		0.91
Balance due Sept. 11		$ 279.57

SOLUTION BY MERCHANTS' RULE

Face of note		$1,000.00
Interest from Mar. 15 to Sept. 11—$1,000 for 180 days		15.00
Maturity value of note, Sept. 11		$1,015.00
First Payment, Mar. 30	$200.00	
Interest from Mar. 30 to Sept. 11—$200.00 for 165 days	2.75	
Second Payment, June 28	6.00	
Interest from June 28 to Sept. 11—$6.00 for 75 days	0.04	
Third payment, Aug. 3	525.00	
Interest from Aug. 3 to Sept. 11—$525.00 for 39 days	1.71	735.50
Balance due Sept. 11		$ 279.50

The difference between the balance as computed by the Merchants' Rule and the balance as computed by the United States Rule becomes larger for greater amounts and for longer periods of time.

PROBLEMS

Solve the following problems first by the United States Rule and then by the Merchants' Rule, and compare the answers:

1. $1,500 with interest at 6%, dated Apr. 1. Payments: June 15, $500; July 10, $250; Oct. 2, $200. Balance due Jan. 15.

2. $600 with interest at 6%, dated Sept. 5. Endorsements: Oct. 1, $120; Nov. 1, $130; Dec. 1, $200. Balance due Dec. 15.

3. \$175 with interest at 5%, dated Aug. 9. Payments: Sept. 9, \$50; Oct. 23, \$100. Balance due Dec. 6.

4. \$3,500 with interest at 6%, dated Jan. 12. Payments: June 16, \$200; Dec. 15, \$2,500; Mar. 17, \$300. Balance due July 16.

5. Consider a loan of P dollars for T years at a rate of r, simple interest. If a payment of P_1 dollars is made on the loan t years after it is made, set up an expression for the balance due at the end of T years according to the United States Rule. (t is less than T; P_1 is less than P.)

6. Use the data of Problem 5, and set up an expression for the balance due at the end of T years according to the Merchants' Rule.

7. Using the results of Problems 5 and 6, find the difference if the balance under the Merchants' Rule is subtracted from the balance under the United States Rule. When is this difference equal to zero?

Exercises

1. A man borrows \$600 for 7 years at 4%. What is the interest?

2. At what rate will \$3,000 produce \$600 interest in 5 years?

3. How long will it take an investment of \$550 to amount to \$700 if placed at 5% simple interest?

4. A note of \$3,500 without interest is due in 5 years. What is the present value of the note if money is worth 2%?

5. What is the true discount on a note for \$780, due in 4 years, money being worth 3%?

6. A note dated May 25, payable in 5 months, without interest, was discounted on July 27 at a bank whose discount rate was 8%. If the proceeds were \$243.76, what was the face of the note?

7. If the exact interest on a principal P for t years at rate r is I_e, show that the ordinary interest I_o is given by the formula $I_o = I_e + I_e/72$.

8. If the exact interest on a certain loan is \$21.30, what would be the ordinary interest?

9. Find the ordinary interest for the approximate number of days on a loan of \$26,000 made on July 14, and due Jan. 17, if money is worth 3%; do the same for the exact number of days.

10. Find the exact interest for the exact number of days on a loan of \$675 made on May 3, and due Mar. 9 following, if money is worth $2\frac{1}{2}$%; do the same for the approximate number of days.

11. Howard Hoose buys a bill of goods and is given an option of paying \$5,000 cash or \$5,200 at the end of 3 months. If he pays cash, at what rate of interest could he consider his \$5,000 invested for the next 3 months?

12. A merchant buys goods for $7,000 due in 3 months with 3% off for cash in 20 days. What is the highest rate at which he could afford to borrow money on the 20th day to pay cash?

13. Johnson buys his wife a fur coat from a furrier who wants $350 at the end of 1 month. If Johnson wishes to pay cash, what should the furrier charge him if money is worth $3\frac{1}{2}\%$?

14. Mr. Jenkins buys a refrigerator for $280 due in 30 days with 2% off for cash. He can borrow money from a credit union for 6% interest. Would he be ahead financially to borrow and take advantage of the discount?

15. Show the division of profits for a partnership business in which A invested $7,000, B invested $5,000, and C invested $4,500 if the net profits for the year amount to $4,000 and, by agreement, each partner is to receive 5% interest on his investment (this interest to be deducted from net profits) and the balance of the profits is to be divided equally.

16. A note for $1,700 is dated May 20, 1961. The interest is 6% from date and the following payments are made on the note:

Sept. 7, 1961	$ 45
Dec. 16, 1961	20
July 8, 1962	140
Nov. 15, 1962	250

By use of the United States Rule, find what amount is due Dec. 30, 1962.

17. A note for $900 with interest at 5% was dated Feb. 3 and had the following endorsements:

Mar. 1	$150
June 3	96
July 8	300
Dec. 20	250

By use of the Merchants' Rule find what amount was due Jan. 17 following.

18. Find the true discount on $220 due in 7 months if money is worth $4\frac{1}{4}\%$.

19. The proceeds of a non-interest-bearing note, discounted at 4% for 3 months, are $400. Find the face value.

20. A note with a maturity value of $75,000 is discounted 10 months before it is due, and the proceeds are $70,000. Find the discount rate.

21. A note with maturity value $600 is discounted on Jan. 17, 1961, at 3%, and the proceeds are $565. When is the note due?

22. What interest rate is equivalent to a discount rate of $8\frac{1}{2}\%$ for 7 months' time? For 1 year's time?

23. A non-interest-bearing note of face value $1,500 is discounted 4 months before it is due at a bank charging 4% discount rate. Find the proceeds. What equivalent rate of simple interest is the bank earning?

24. Smith gives Brown his 90-day note for $230 bearing interest at 3% on June 26. Brown discounts the note on July 7 at the First National Bank, which charges 5% simple discount. Find the proceeds.

25. The Fairview Department Store discounted a 100-day note for $30,000 bearing interest at $2\frac{1}{2}$% and dated Aug. 22, at their bank on Sept. 1. The bank's discount rate was 1%. Find the proceeds.

26. Find the proceeds of the following note if discounted on Mar. 6, 1956, at a bank charging 2% discount rate:

Bellefonte, Feb. 8, 1956

Six months after date I promise to pay to W. O. Rogers $930 together with interest from date at $3\frac{1}{2}$%.

Signed: Paul Beardslee

27. Mason owes Jones $10,000. He gives him a 45-day non-interest-bearing note from which Jones gets his money by discounting it immediately at a bank charging 4% discount rate. What is the face of the note?

28. A merchant wishes to pay cash and receive a 5% cash discount on a bill for $1,000. How much must he borrow from a bank on a 120-day note in order to do so if the bank discounts his note at $4\frac{1}{2}$%?

29. Find the present worth of $312 due in 6 months if money is worth 4%.

30. How long will it take $144 to amount to $145.50 at $2\frac{1}{2}$% simple interest?

31. Find the exact interest on $292 for 360 days at 4%.

32. What bank discount rate is equivalent to an interest rate of $3\frac{1}{2}$% if the discount is for a term of 10 years?

33. A note for $500 bearing interest at 6% is due in 6 months. A first payment of $200 is made at the end of 3 months. Find, by the United States Rule, the balance due after the first payment is made.

34. A S. Fite can buy a car for $1,500 due at the end of 3 months or at 10% cash discount if paid within 1 month. He elects to take advantage of the cash discount but must borrow money to do so. At what rate should he borrow at the end of 1 month so that his method of payment is equivalent to the 3-month method?

35. A refrigerator sells for $290 cash or $80 down and $13.50 a month for 18 months. Find the rate of interest paid by a customer using the installment plan.

36. An electric stove sells for $310 cash or $80 down and $13 a month for 18 months, with the nineteenth payment being $7. Find the rate of interest paid by a customer using the installment plan.

Self-Test

Time: 50 Minutes

(Score 5 points for each problem correctly solved)

1. Find the exact simple interest on $600 for 144 days at 5%.

2. What interest rate is earned by a bank discounting a note for 2 years if 1% discount is charged?

3. How long will it take for $900 to amount to $967.50 at 3% simple interest?

4. Find the exact number of days between Jan. 4, 1963, and Aug. 9, 1963.

5. Find the exact number of days between Mar. 23, 1963, and July 5, 1963.

6. Find the exact number of days between Veterans Day and Christmas.

7–9. Find the approximate number of days between each of the pairs of dates given in Problems 4 to 6.

10. Find the true discount on $800 due in 3 months if money is worth 4%.

11. Find the present value of a note for $500 due in 9 months if money is worth 4%.

12. A set of books sells for $100 cash or $20 down and $3.20 a week for 26 weeks, and a payment of $1.80 on the 27th week. Find the installment interest rate.

13, 14. A debt of $900 contracted on Apr. 15 bears interest at 2%. A payment of $400 is made on Aug. 3. Find the amount due on Nov. 1 by the Merchants' Rule; by the United States Rule.

15, 16. The Berkley Square Department Store discounts at a bank a non-interest-bearing note for $1,000 due in 6 months. If the bank discount rate is 2%, what is the bank discount? The proceeds of the note?

17, 18. A note dated Jan. 5, for $1,600, and bearing interest at 4%, falling due on Apr. 5, was cashed at a bank on the former date. If the bank discount rate was 1%, what was the maturity of the note? The bank discount?

19, 20. A note for $990, payable on July 1, bearing interest at 3%, was signed by A. C. Miller on Mar. 3. If the bank discount rate was $1\frac{1}{2}$%, what amount was due on July 1, and what were the proceeds of the note if discounted on May 2?

Chapter VI

FURTHER ALGEBRAIC OPERATIONS

Introduction. In a previous chapter, we extended the number system of arithmetic by the introduction of negative numbers and proceeded to show how the simple operations of addition, subtraction, multiplication, and division were carried out when symbols were used to represent quantities. Rules for these operations were developed, and opportunity for practice in their use was provided. In that chapter, the student became acquainted with the meaning of algebra and had a limited view of some of the practical applications of algebra in the solution of problems.

In the present chapter, we shall continue our treatment of algebraic operations as further aid in the solving of business problems. In particular, we shall consider a few operations involving polynomials and expressions containing fractions; and we shall extend the definition of exponents to include fractional exponents, negative exponents, and radicals. Meaning will be given to the exponent zero, and practice will be afforded in the use of exponents as preparation for the understanding of logarithms and the solution of certain types of equations encountered later in problems of finance and other business applications. The chapter will close with a discussion of square root, rectangular coordinates, and the construction of formula graphs.

64. Addition and Subtraction of Polynomials. Algebraic expressions containing more than one term (see Section 20) are added or subtracted by arranging like terms in the same column and then combining them as was done in the case of single terms.

Illustrations:

1. Add: $M_1 + M_2$, $2M_1 + M_3$, $4M_2 - 6M_3$.

Arrange the expressions so that like terms are in a column.

$$\begin{array}{lrcrcr}
 & M_1 & + & M_2 & & \\
 & 2M_1 & & & + & M_3 \\
 & & & 4M_2 & - & 6M_3 \\ \cline{2-6}
\text{Adding,} & 3M_1 & + & 5M_2 & - & 5M_3
\end{array}$$

2. Subtract $4a - 2b$ from $9a + 5b - 3c$.

Arrange the expressions so that like terms are in a column, with the subtrahend beneath the minuend.

$$\begin{array}{ll} & 9a + 5b - 3c \\ & \underline{4a - 2b\phantom{{}-3c}} \\ \text{Subtracting,} & 5a + 7b - 3c \end{array}$$

PROBLEMS

Add the following polynomial expressions:

1. $5x + 7y,\ 3x - y,\ x - 3y,$
2. $a - 6b,\ 4a - b,\ a + b,$
3. $15x^2y - 4y,\ -4y - 6x^2y,\ -x^2y + y,$
4. $6x^3yz,\ 3y^2 - x^3yz,\ -4y^2 - 7x^3yz,$
5. $6x^2,\ 7y^2,\ 7x^2 - 6y^2,\ -7x^2 - 7y^2,$
6. $2a + 4b - 3c + 7,\ 3a - 2b + 4c - 11,\ -6a + 7b - c + 9,$
7. $6x^2y - 4xy^2 + 6,\ 9 - x^2y,\ 4xy^2 - 12x^2y,$
8. $6c - 5d + 7,\ 3a - 2b + 6,\ 4e - 8f - 13,$
9. $y^3 - y^2 + y - 8,\ 2y^3 - 6y + 5,\ 17 - 9y^2,\ 3y^3 + 6y + 100,$
10. $4.7x + 2.9y,\ 0.6y - 1.5x,\ -0.2y - 0.9x.$

Subtract the following algebraic expressions:

11. $2a$ from $5a$,
12. $-3r$ from $2r$,
13. $4i$ from $-6i$,
14. $2x^2y$ from $9x^2y$,
15. $3a + 4b$ from $5a + 5b$,
16. $9P - S$ from $10P + 3S$,
17. $x + 5y - 4w$ from $3x - 8y + w$,
18. $a + 2b - 3c$ from $3a - 2b + c$,
19. $4x^3 - 5y^2 + 7$ from $3x^3 + 2y^2$,
20. $x^3 + 5x^2 - 10$ from $x^4 - 2x^3 + 2x$,
21. $9rt^2 - 4r + 6t - 7$ from 12,
22. $6x^2yz + 2xy - 7yz$ from $15yz - 9xy + x^2yz$,
23. $\frac{1}{3}P + \frac{2}{5}Q - \frac{3}{4}R$ from $\frac{3}{8}R - \frac{5}{6}P + \frac{3}{10}Q$,
24. $\frac{9}{7}S + \frac{2}{3}P - 9D$ from $D - \frac{4}{9}P + \frac{3}{14}S$.
25. Subtract $a - b + 3c$ from the sum of $2a + 2b - 6c$ and $-4a - 7b + 2c$.

26. Subtract $7 - 2y + 3x$ from the sum of $x + 6y - 10$ and $3y - 4x - 12$.

27. Subtract $xy - 3x - 2y$ from the sum of $10 + 2xy$ and $7x - 13$.

65. Multiplication of Polynomials. As a result of the fundamental assumptions of algebra (see Section 22), the following rule for multiplication of polynomials may be stated:

To multiply polynomials, multiply each term of one polynomial (the multiplicand) by each term of the other (the multiplier), and add the results.

The method for carrying out the multiplication of polynomials is illustrated below:

Illustrations:

1. Multiply $3x - 4y$ by $2x + 5y$.

$$\begin{array}{lrl}
 & \text{Multiplicand} = & 3x - 4y \\
 & \text{Multiplier} = & \underline{2x + 5y} \\
 & 2x \text{ times } (3x - 4y) = & 6x^2 - 8xy \\
 & 5y \text{ times } (3x - 4y) = & \underline{\qquad\quad 15xy - 20y^2} \\
\text{Adding,} & \text{Product} = & 6x^2 + 7xy - 20y^2
\end{array}$$

2. Multiply $x^2 - 4x + 5$ by $2x - 3$.

$$\begin{array}{lrl}
 & \text{Multiplicand} = & x^2 - 4x + 5 \\
 & \text{Multiplier} = & \underline{2x - 3} \\
 & 2x \text{ times } (x^2 - 4x + 5) = & 2x^3 - 8x^2 + 10x \\
 & -3 \text{ times } (x^2 - 4x + 5) = & \underline{\qquad\ - 3x^2 + 12x - 15} \\
\text{Adding,} & \text{Product} = & 2x^3 - 11x^2 + 22x - 15
\end{array}$$

PROBLEMS

Multiply:

1. $(a - b)(a + b)$,
2. $(c + d)(c + 2d)$,
3. $(x + 2y)(x + y)$,
4. $(x - y)(x - y)$,
5. $(2r + d)(2r + d)$,
6. $(x - w)(x + w)$,
7. $(F - 3)(F + 4)$,
8. $(S + 2P)(S - P)$,
9. $(C + 3)(C - 2)$,
10. $(p + q)(p - q)$,
11. $(a + L)(a + L)$,
12. $(3x - 7y)(2x + y)$,

13. $(2c + d)(9c - 4d)$,
14. $(3x^2 - 2z)(3x^2 + 2z)$,
15. $(5a^2 + 2b)(4a^2 - b)$,
16. $(x^4 + y^4)(x - y)$,
17. $(i^3 - 8r^2)(i - 2r)$,
18. $(u^2 + v^2)(u^2 - v^2)$,
19. $(x - y)(x^2 + xy + y^2)$,
20. $(F + D)(F^2 - FD + D^2)$,
21. $(c + 3d)(c^2 - 2d + d)$,
22. $(F + 3)(2F^2 - 8Fp + 9)$,
23. $(a + b)(a^2 + 2ab + b^2)$,
24. $(x - y)(y^2 - 12x + 36)$,
25. $(x^3 + y^3)(a - 3x + 2y)$,
26. $(z - 2w)(z^2 + 2zw + 4w^2)$,
27. $(m - n)(m^4 + m^3n + m^2n^2 + mn^3 + n^4)$,
28. $(p + q)(p^4 - p^3q + p^2q^2 - pq^3 + q^4)$,
29. $(a + b - ab)(a + b + ab)$,
30. $(x^2 + xy + y^2)(x^2 - xy + y^2)$,
31. $(3a^2 + a - 1)(2a^2 - 3a + 2)$,
32. $(a + b + c)(a - b - c)$, 33. $(1 + x + y)(1 + x - 3y)$,
34. $(9r^2 - 3i^2j + 2j^2)(8r^2 - 5j^2 - 2i^2j)$,
35. $(P + S)(P + S)(P + S)$, 36. $(p - rt)(1 + r)(p + t)$,
37. $(0.7p^2 - 0.2p + 0.7)(0.35p^2 + 0.6p - 0.84)$,
38. $(1 - q)(2 - q)(3 - q)(4 - q)$.

66. Factoring Polynomials. The problem of expressing polynomials as the product of factors is complicated because of the great variety of forms of polynomials. In Section 29, we saw that the monomial factors of a polynomial could readily be obtained by inspection; for example, the factors of $PRT + 100P$ are the monomial factor P and the binomial factor $RT + 100$, that is, $PRT + 100P = P(RT + 100)$. There is no direct process for obtaining the factors of any given polynomial expression, so that we must depend upon our ability to recognize a given expression as having arisen from multiplication of certain factors.

If the polynomial is the difference between the squares of two numbers, the sum of the numbers is one factor and the difference of the numbers is the other. Thus, $x^2 - 25$ may be written as $(x + 5)(x - 5)$. Justification for this easy rule lies in the fact that multiplying $(a + b)(a - b)$ gives $a^2 - b^2$, and we have simply replaced a by x and b by 5.

Factoring is usually limited to finding the factors of polynomials with rational coefficients, and we try to find factors that contain only rational numbers. Often, very complicated expressions may be written in factor form if the worker can hit upon suitable combinations which, when multiplied, will yield the given expressions. The following illustrations will indicate how certain trinomials (polynomials containing three terms) may be factored.

Illustrations:

$$\begin{aligned} x^2 + 7x + 12 &= (x + 3)(x + 4), \\ x^2 - x - 12 &= (x + 3)(x - 4), \\ 2x^2 + 9x + 10 &= (2x + 5)(x + 2), \\ 12x^2 + 26x + 10 &= (4x + 2)(3x + 5), \\ 10x^2 + 11x - 6 &= (5x - 2)(2x + 3). \end{aligned}$$

PROBLEMS

Factor:

1. $3abc - 27ab$,
2. $2r - 4rt + 3r^2$,
3. $x^2 - y^2$,
4. $16 - a^2$,
5. $x^2 + 6x + 9$,
6. $42 - G - G^2$,
7. $a^2b^2 - 25m^2n^2$,
8. $R^2 + 4R - 21$,
9. $2a^2 + 5a + 2$,
10. $AR^3 - AR$,
11. $64z^2 - 48zw + 9w^2$,
12. $D^2 + 0.5D + 0.06$,
13. $50F^2 - 72N^2$,
14. $10x^2 - 26xy + 12y^2$,
15. $3ca^4 - 3b^4c$,
16. $8r^2 - 15rs - 2s^2$,
17. $30a^2 + 17ap - 42p^2$,
18. $12p^2 + 40p - 7$,
19. $(p + q)^2 - 1$,
20. $625w^2 - 100wv + 4v^2$,
21. $c^2d - d^2c$,
22. $16x^2 + 102x - 13$,
23. $2z^2y - 14zy^2 + 8y$,
24. $a^2e^4 - b^2f^4$,
25. $(1 + r)^2 - (1.03)^2$,
26. $1 + 2rt + r^2t^2$,
27. $Adt^n + t^{n-2}$,
28. $y^2 + 2xy + x^2 - z^2$,
29. $144pq^4 + 72pq^2 + 9p$,
30. $81s^2p^2 + 36spr + 4r^2$.

67. Addition and Subtraction of Fractions. Everything that was learned about handling fractions in arithmetic may be carried over into operating with fractions in algebra. Fractions may be added and subtracted, multiplied and divided. Just as in arithmetic, we shall need to know how to get a least common denominator when adding or subtracting; and we shall need to remember that when we divide one fraction by another, we invert the divisor and multiply. One basic principle we must keep in mind is that the value of a fraction is unchanged if the numerator and the denominator are both multiplied by the same quantity (zero excepted). The following illustrations will serve to demonstrate how the least common denominator is found, how to change fractions to equivalent fractions with the least common denominator, and finally how to combine fractions by addition and subtraction.

Illustrations:

1. Find the least common denominator for

$$\frac{3}{a^4}, \qquad \frac{7}{ax^2y^3}, \qquad \frac{-5}{2a^3}.$$

First, write the prime factors of each denominator.

$$\begin{aligned} a^4 &= a \cdot a \cdot a \cdot a, \\ ax^2y^3 &= a \cdot x \cdot x \cdot y \cdot y \cdot y, \\ 2a^3 &= 2 \cdot a \cdot a \cdot a. \end{aligned}$$

Then write the product of all the different factors, using each factor the greatest number of times found in any one denominator.

$$2 \cdot a \cdot a \cdot a \cdot a \cdot x \cdot x \cdot y \cdot y \cdot y = 2a^4x^2y^3 \qquad \text{(LCD)}.$$

The student will soon discover that the LCD can usually be found by inspection.

2. Change the fractions of the preceding illustration to equivalent fractions having the LCD.

When the LCD, $2a^4x^2y^3$, is divided by the denominator of the first fraction (a^4) we obtain $2x^2y^3$, which must now be multiplied by the numerator (3) of that fraction. As our new numerator we thus obtain $3 \cdot 2x^2y^3$, or $6x^2y^3$. Hence, for the first fraction we have

$$\frac{3}{a^4} = \frac{3 \cdot 2x^2y^3}{2a^4x^2y^3} = \frac{6x^2y^3}{2a^4x^2y^3}.$$

Likewise, for the other two fractions we obtain

$$\frac{7}{ax^2y^3} = \frac{7 \cdot 2a^3}{2a^4x^2y^3} = \frac{14a^3}{2a^4x^2y^3},$$

and

$$\frac{-5}{2a^3} = \frac{-5 \cdot ax^2y^3}{2a^4x^2y^3} = \frac{-5ax^2y^3}{2a^4x^2y^3}.$$

3. Combine the three fractions of Illustration 1 into one fraction by writing the numerators in succession over the common denominator.

The three given fractions written in terms of their least common denominator are, as found in Illustration 2,

$$\frac{6x^2y^3}{2a^4x^2y^3}, \qquad \frac{14a^3}{2a^4x^2y^3}, \qquad \text{and} \qquad \frac{-5ax^2y^3}{2a^4x^2y^3}.$$

Hence, $$\frac{3}{a^4} + \frac{7}{ax^2y^3} + \frac{-5}{2a^3} = \frac{6x^2y^3 + 14a^3 - 5ax^2y^3}{2a^4x^2y^3}.$$

4. Add $\frac{-a}{b} - \frac{a}{b} + \frac{a}{-b}$.

Since both numerator and denominator of a fraction can be multiplied by the same nonzero quantity without changing the value of the fraction, it is seen that

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

Both numerator and denominator of the fraction $\frac{-a}{b}$ were multiplied by (-1) to obtain $\frac{a}{-b}$; and the last equality was found by the rule for dividing a positive number by a negative one. Thus, we can write our original problem as

$$-\frac{a}{b} - \frac{a}{b} - \frac{a}{b} = 3\left(-\frac{a}{b}\right) = -3 \cdot \frac{a}{b} \quad \text{or} \quad \frac{-3a}{b} \quad \text{or} \quad -\frac{3a}{b}.$$

It is seen that the rule for changing the signs of a fraction can be formulated as follows:

Of the three possible signs in a fraction (in numerator, in denominator, and in front of the fraction), any two can be changed without changing the value of the fraction.

5. Add $\frac{3}{a^2 - b^2} - \frac{2}{b - a}$.

The LCD would be $a^2 - b^2$ if $b - a$ could be written as $a - b$. This can be done by the rule for changing the signs of a fraction. Thus, since

$$-\frac{2}{b - a} = +\frac{2}{a - b},$$

the problem can be written

$$\frac{3}{a^2 - b^2} + \frac{2}{a - b}.$$

Changing into equivalent fractions having the LCD and adding, we have

$$\frac{3}{a^2 - b^2} + \frac{2(a + b)}{a^2 - b^2} = \frac{3 + 2a + 2b}{a^2 - b^2}.$$

PROBLEMS

Perform the indicated operations:

1. $\frac{24}{1 + r} - \frac{12}{(1 + r)^2}$,
2. $\frac{na}{2} + \frac{4D}{K}$,
3. $\frac{2ab}{x^2y} - \frac{3a}{xy^2}$,
4. $\frac{4a}{5} - \frac{5}{8a}$,
5. $\frac{4W}{7} - 3W - \frac{3W}{4}$,
6. $\frac{6}{F} - \frac{2}{F^2} + \frac{3}{5}$,
7. $\frac{2x}{9} + \frac{3}{7x^2} + \frac{5}{3x}$,
8. $\frac{4a}{(2a - 3)} - \frac{5a}{(a + 2)}$,
9. $\frac{P}{P + 1} - 0.04$,
10. $\frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x}$,
11. $\frac{2P}{3R^2S} - \frac{P - S}{3R} + \frac{S}{2R}$,
12. $\frac{x}{x^2 - y^2} + \frac{y}{x - y} - \frac{y}{x + y}$,
13. $6 - \frac{p + q}{3q} + \frac{p - q}{2p}$,
14. $\frac{a}{a^2 + 2ab + b^2} + \frac{b}{a + b} - \frac{(a + b)}{2}$,
15. $\frac{x + y}{x^2 - xy - 12y^2} - \frac{x}{(x - 4y)^2} + \frac{y}{x + 3y}$,
16. $\frac{3 - 2a}{2a - ba} - \frac{3}{b - 2}$,
17. $\frac{3}{2i} + \frac{k + 3}{ik - 2i} + \frac{1}{k - 2}$,
18. $\frac{1}{A} - \frac{1}{B} + \frac{1}{AB} + \frac{A - B}{AB}$,
19. $\frac{A}{Adt - A} - \frac{d + 1}{d^2t^2 - 1}$,
20. $\frac{3x^2 + 4xy}{2x^2 + xy - 6y^2} - \frac{3x}{2x + 4y}$.

68. Multiplication and Division of Fractions. To multiply fractions, we multiply their numerators for the numerator of their product and multiply their denominators for the denominator of the product. Factors common to numerator and denominator may be canceled out either before or after the multiplication is performed.

Illustrations:

$$\frac{a^4}{b^3}\cdot\frac{b}{a^2y}=\frac{a^2}{b^2y},$$

$$\frac{(2x+3)^2}{12}\cdot\frac{4}{(2x+3)}=\frac{4(2x+3)^2}{12(2x+3)}=\frac{(2x+3)}{3}.$$

To divide one fraction by another, invert the divisor and then multiply.

Illustration:

$$\frac{3}{(1+x)^2}\div\frac{x-1}{x+1}=\frac{3}{(1+x)^2}\cdot\frac{x+1}{x-1}=\frac{3}{(1+x)(x-1)}=\frac{3}{x^2-1}.$$

PROBLEMS

Perform the indicated operations, and simplify:

1. $\dfrac{a^4b^3c^5}{b^2c^3y}\div\dfrac{ax^2y^4}{a^2c}$,

2. $\dfrac{a^2-b^2}{3xy}\cdot\dfrac{24x^2}{y(a-b)}$,

3. $\dfrac{4m^2n^3}{3rs}\cdot\dfrac{15r^2s^2}{8mn^4}$.

4. $\dfrac{5ab}{a-b}\div\dfrac{3ax}{a+b}$,

5. $\dfrac{10a^5z^4w}{3ac}\div\dfrac{2z^3w^2}{6ad}$,

6. $\dfrac{\frac{1}{3}+\frac{1}{2}}{\frac{5}{6}}$,

7. $\dfrac{x-\dfrac{1}{y}}{y-\dfrac{1}{x}}$,

8. $\dfrac{n}{1-\dfrac{1}{n+1}}$,

9. $\left(\dfrac{7a^2}{b}\div b^2\right)\cdot\dfrac{b^2}{21}$,

10. $\dfrac{Prt}{100+k}\div\dfrac{P}{100}$,

11. $\dfrac{x^2-y^2}{3x}\cdot\dfrac{6x^2-9x}{x-y}$,

12. $\dfrac{a+b}{a}\cdot\dfrac{a+1}{a-b}$,

13. $\dfrac{\frac{1}{x}}{x}$,

14. $\dfrac{1}{\frac{1}{x}}$,

15. $\dfrac{3x^2 - 8x + 4}{x + 3} \div \dfrac{x^2 - 4x + 4}{x^2 - 9}$,

16. $\dfrac{R - S}{R + S} \div \dfrac{R + S}{R - S}$,

17. $\dfrac{i^2 - r^2}{i^3r^4} \cdot \dfrac{ir}{i - r}$,

18. $\dfrac{\frac{1}{n} - \frac{1}{n}}{n}$,

19. $(q^2 - 2q + qp) \div \dfrac{q}{p}$,

20. $\left(\dfrac{xy - x}{xy + y} \div \dfrac{y - 1}{x + 1}\right) \cdot \dfrac{y}{x}$,

21. $\dfrac{A}{(1 + r)^3} \cdot (1 + r)$,

22. $\dfrac{Pdt}{(1 + d)^3} \div \dfrac{1}{1 + d}$,

23. $\dfrac{C^2 - P^2}{4} \cdot \dfrac{800}{C - P}$,

24. $\dfrac{-19p^5q^7}{95q} \div \dfrac{r^2s}{rs^2}$,

25. $\dfrac{44i^3}{r^3d^5} \div \dfrac{-11i}{r^3d^7}$,

26. $\dfrac{38 + 13r - 3r^2}{27r} \cdot \dfrac{3r^2}{3r - 19}$,

27. $\dfrac{2k^2}{5k^2 - mk - 6m^2} \cdot \dfrac{6m - 5k}{6mk}$,

28. $\dfrac{1}{3}\left(x - \dfrac{x}{2}\right)\left(y - \dfrac{y}{2}\right) \div \dfrac{y}{12}$.

69. Fractional Exponents. In Chapter II our discussion about exponents was confined to positive integral exponents only, and we used these exponents in multiplication, division, and raising to powers. The fundamental laws for positive integral exponents may be summarized as follows:

1. Law of multiplication: $a^m \cdot a^n = a^{m+n}$.
2. Law of division: $a^m \div a^n = a^{m-n}$ $(a \neq 0,\ m > n)$.
3. Law for power of a power: $(a^m)^n = a^{mn}$.
4. Law for power of a product: $(abc \cdots)^m = a^mb^mc^m \cdots$.

We may extend the definition of exponents to include fractional exponents by defining fractional exponents in such a way that the laws for positive integers also hold for fractional exponents. Accordingly, we define

$$a^{\frac{1}{2}} = \sqrt{a},$$

$$a^{\frac{1}{3}} = \sqrt[3]{a},$$

and so forth, and, in general,

$$a^{1/n} = \sqrt[n]{a},$$

and

$$a^{m/n} = (\sqrt[n]{a})^m,$$

provided that a is positive when n is even. The symbol $\sqrt[n]{\ }$, known as a *radical*, denotes the root that is to be taken. The number or expression under the radical sign is called the *radicand*, and the index of the radical specifies the root to be taken and the *order* of the radical. Thus, $\sqrt[3]{27}$ means "the cube root of 27," $\sqrt[6]{64}$ means "the sixth root of 64," and so forth. For the square root it is customary to omit the index 2 and write the radical sign $\sqrt{\ }$. When there are two real roots, the radical without a sign prefixed indicates the positive real root. We say, therefore, that $a^{1/n}$ means the nth root of a; and that $a^{m/n}$ means the mth power of the nth root of a, or the nth root of the mth power of a.

Illustrations:

1. $9^{\frac{1}{2}} = \sqrt{9} = 3,$ 3. $(-64)^{\frac{1}{3}} = \sqrt[3]{-64} = -4,$

2. $9^{\frac{3}{2}} = (\sqrt{9})^3 = 3^3 = 27,$

4. $(8a^3)^{\frac{2}{3}} = (\sqrt[3]{8a^3})^2 = (2a)^2 = 4a^2,$ or $\sqrt[3]{(8a^3)^2} = \sqrt[3]{64a^6} = 4a^2,$

5. $16^{\frac{1}{2}} \cdot 16^{\frac{1}{2}} = 16^{\frac{1}{2}+\frac{1}{2}} = 16^1 = 16,$

6. $27^{\frac{2}{3}} \div 27^{\frac{1}{3}} = 27^{\frac{2}{3}-\frac{1}{3}} = 27^{\frac{1}{3}} = 3,$

7. $(25^{\frac{3}{2}})^2 = 25^{\frac{3}{2}\cdot 2} = 25^3,$ 8. $(4^{\frac{1}{2}} \cdot 9^{\frac{3}{2}})^4 = 4^{\frac{1}{2}\cdot 4} \cdot 9^{\frac{3}{2}\cdot 4} = 4^2 \cdot 9^6,$

9. $(27x^3y^6)^{\frac{2}{3}} = (\sqrt[3]{27x^3y^6})^2 = (3xy^2)^2 = 9x^2y^4,$ or $\sqrt[3]{(27x^3y^6)^2} = \sqrt[3]{729x^6y^{12}} = 9x^2y^4,$

10. $(a^{\frac{1}{3}}b^2)^{\frac{1}{4}} \cdot (ab)^{\frac{2}{3}} = a^{\frac{1}{12}}b^{\frac{1}{2}} \cdot a^{\frac{2}{3}}b^{\frac{2}{3}} = a^{\frac{1}{12}+\frac{2}{3}} \cdot b^{\frac{1}{2}+\frac{2}{3}} = a^{\frac{3}{4}}b^{\frac{7}{6}}.$

PROBLEMS

1. Express with radical signs:

(a) $(50)^{\frac{1}{2}}$, (f) $(18)^{\frac{1}{3}} \cdot (16)^{\frac{1}{3}}$, (k) $(6x^2y^3)^{\frac{2}{5}}$,
(b) $(1.02)^{\frac{1}{3}}$, (g) $(1 + i)^{\frac{1}{4}}$, (l) $14xy^{\frac{3}{4}}$,
(c) $(16x)^{\frac{1}{3}}$, (h) $(12)^{\frac{2}{3}}$, (m) $14(xy)^{\frac{3}{4}}$,
(d) $16x^{\frac{1}{4}}$, (i) $7(5r)^{\frac{2}{5}}$, (n) $2R^{\frac{1}{5}}$,
(e) $(xy)^{\frac{1}{5}}$, (j) $(R + S)^{\frac{3}{4}}$, (o) $(18p^3q^2)^{\frac{1}{6}} \cdot (1.04)^{\frac{2}{3}}$.

2. Express with fractional exponents:

(a) $3a\sqrt{b}$, (f) $\sqrt[n]{(p + q)^k}$, (k) $\sqrt[3]{a^3 - x^3}$,
(b) $2\sqrt[3]{b^2}$, (g) $\sqrt{7^3a}$, (l) $3C\sqrt{C^2 - 5^2}$,
(c) $-9m^4\sqrt{m}$, (h) $\sqrt[5]{i^5r^3}$, (m) $\sqrt{S^{\frac{1}{2}}}$,
(d) $\sqrt[3]{m^2n^2}$, (i) $\sqrt[3]{(RS)^2} \cdot \sqrt[4]{T^3}$, (n) $\sqrt[3]{p^{\frac{1}{2}}q^{\frac{1}{3}}}$,
(e) $\sqrt[3]{(1 + r)^5}$, (j) $\sqrt{3(1 + i)}$, (o) $\sqrt[4]{d^{12}g^{\frac{3}{5}}}$,

3. Simplify the following expressions. Be sure to apply the fundamental laws for multiplying, dividing, and raising to a power where fractional exponents are met as well as to the already familiar positive integral exponents.

(a) $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{6}}$,

(b) $(\frac{3}{4})^5 \div (\frac{3}{4})^{\frac{5}{2}}$,

(c) $(3s)^{\frac{1}{3}} \cdot (3s)^{\frac{2}{3}}$,

(d) $[(4r)^2]^{\frac{1}{2}}$,

(e) $(27^{\frac{2}{3}} - 5)^{\frac{1}{2}}$,

(f) $(\sqrt[3]{-0.001})^4$,

(g) $(x^2y^4z^3)^{\frac{1}{2}} \cdot (xyz^2)^{\frac{3}{4}}$,

(h) $(16z^3w)^{\frac{3}{4}} \div (27z^4)^{\frac{1}{3}}$,

(i) $(r^3S)^{\frac{1}{6}} \div (rS^{\frac{1}{2}})^{\frac{1}{6}}$,

(j) $\dfrac{81^{\frac{3}{4}} \cdot 8^{\frac{2}{3}} \cdot (\frac{1}{4})^{\frac{1}{2}}}{9^{\frac{1}{2}} \cdot 64^{\frac{1}{3}}}$.

(k) $[\sqrt[3]{(-2a)^6} - 4a^3]^{\frac{1}{2}}$,

(l) $(1 + 0.05)^{\frac{3}{4}} \cdot \sqrt[4]{1.05}$,

(m) $\sqrt{(1 + r)^3} \cdot \sqrt{1 + r}$,

(n) $(1.03)^{15} \div [(1.03)^{\frac{1}{2}}]^8$,

(o) $(1.04)^{\frac{7}{2}} \cdot \dfrac{1}{(1.04)^{\frac{3}{2}}}$,

(p) $P(1 + r)^n \cdot \dfrac{A}{(1 + r)^2}$,

(q) $\dfrac{2 - 2(\frac{1}{3})^4}{1 - \frac{1}{3}}$,

(r) $\dfrac{3 - 3\,(\sqrt{2})^8}{-5}$,

(s) $60\left[\dfrac{0.04 - (1 + 0.04)^2}{0.04}\right]$,

(t) $\dfrac{32^{\frac{3}{5}} \cdot 1000^{\frac{1}{3}} \cdot \sqrt{(9)^3}}{27\,\sqrt[3]{(4.7)^3 \cdot 0} + 400}$.

70. Zero Exponent. By assuming that the laws of exponents hold for all positive exponents and zero, we may find a meaning for a^0, where a is not equal to zero. According to the law for multiplication we have that

$$a^m \cdot a^n = a^{m+n}$$

If $m = 0$, we have

$$a^0 \cdot a^n = a^{0+n} = a^n.$$

Hence, since the multiplier a^0 leaves a^n unchanged, we write $a^0 = 1$.

We may also show that any number other than zero affected by the exponent 0 equals 1 by writing

$$a^m \div a^n = a^{m-n} \qquad (m \text{ greater than } n).$$

When $m = n$, we have

$$a^n \div a^n = a^{n-n} = a^0.$$

But $\qquad a^n \div a^n = 1,$

Therefore, $\qquad a^0 = 1 \qquad (a \neq 0).$

Illustrations:

1. $2^0 = 1$,
2. $(1.06)^0 = 1$,
3. $(-7)^0 = 1$,
4. $(3x)^0 = 1$,
5. $3x^0 = 3 \cdot 1 = 3$,
6. $K^0 + (1 + r)^0 = 1 + 1 = 2$.

PROBLEMS

Find the value of:

1. $4a^0 + 64$,
2. $(4a)^0 + 64$,
3. $(1 + i)^0(1 + i)^2(1 + i)^4$,
4. $(p + q)^6 \div (p + q)^0 - 1^0$,
5. $2a^0(2a)^0$,
6. $3x^2y^3 \cdot x^0 \cdot 0$,
7. $(RST)^0 \div R^0S^0T^0$,
8. $(x^3y^2)^{\frac{1}{2}} \cdot (x^2y^3)^0$,
9. $[(1.08)^3(1.06)^0]^2$,
10. $(3pq)^0 \cdot 3pq^0$,
11. $(6^0)^0$,
12. $(472.65)^0$.

71. Negative Exponents. The definition of exponents may also be broadened so as to give meaning to negative exponents. In order for a^{-n} to obey the law of multiplication, we must have

$$a^{-n} \cdot a^n = a^{-n+n} = a^0 = 1 \qquad (n \text{ positive, } a \neq 0).$$

That is to say, a^{-n} and a^n must be reciprocals, since $1/a^n$ is the only number that will multiply a^n to give 1.

Hence, we write $$a^n = \frac{1}{a^{-n}}.$$

We see, therefore, that a number raised to a negative power is equal to the reciprocal of the same number raised to the corresponding positive power if the multiplication law of exponents is to be satisfied. This definition of a negative exponent will satisfy all the laws of exponents we have studied.

Illustrations:

1. $4^{-2} = \frac{1}{4^2} = \frac{1}{16}$,
2. $(1 + i)^{-3} = \frac{1}{(1 + i)^3}$,
3. $\frac{1}{9^{-2}} = 9^2$,
4. $8^{-\frac{2}{3}} = \frac{1}{8^{\frac{2}{3}}} = \frac{1}{2^2} = \frac{1}{4}$.

PROBLEMS

1. Give the value of: $(-27)^{\frac{2}{3}}$, $(\frac{3}{4})^{-3}$, $(81)^{-\frac{3}{4}}$, $(1.2)^{-2}$.

2. Write with positive exponents in simplified form:

(a) $(1.06)^{-5}$,
(b) $(1.04)^{-\frac{1}{2}}$,
(c) $(1+r)^{-15}$,
(d) $(p+q)^{-n}$,
(e) $\dfrac{1}{(a+b)^{-3}}$,
(f) $r^{-3}+t^{-5}$,
(g) $P(1+R)^{-11}$,
(h) v^{-n},
(i) $\dfrac{x^{-3}y^{2}z^{-1}}{y^{-2}x^{4}}$,
(j) $a^{-3}+b^{-3}$,
(k) $p^{-2}q^{2} \div (1+r)^{-2}$,
(l) $(\frac{6}{7})^{-\frac{2}{3}}$,
(m) $(\frac{1}{5})^{-\frac{1}{2}}$,
(n) $6^{-1}+\dfrac{1}{6^{-1}}$,
(o) $100(1+r)^{-4}$,
(p) $670(1.03)^{-7}$.

Simplify and leave your answer free from negative exponents:

3. $(x^{2}y^{3})^{-\frac{1}{3}}(x^{3}y^{2})^{\frac{1}{2}}$,
4. $(r^{\frac{3}{4}}s^{-\frac{1}{3}})^{-2}(rs^{\frac{1}{3}})^{3}$,
5. $\dfrac{a^{5}b^{-2}c^{-3}}{a^{-2}b^{3}c^{-1}}$,
6. $\dfrac{x^{-\frac{3}{4}}z^{2}}{x^{\frac{1}{4}}z^{3}}$,
7. $\dfrac{(3R^{-2}S^{-\frac{2}{3}})^{\frac{1}{2}}}{(3R^{0}S^{\frac{1}{3}})^{\frac{2}{3}}}$,
8. $\dfrac{(C^{-2}S^{4}T^{0})^{\frac{1}{2}}}{CST}$,
9. $(x^{-2}y^{\frac{3}{4}})(xy)^{-3} \cdot xy$,
10. $(abc)(abc)^{\frac{1}{3}}u^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$,
11. $(i^{-1}j^{2}k^{n})^{\frac{1}{2}}(ij^{-\frac{1}{2}}k)^{-1}$,
12. $x^{-\frac{1}{2}}y^{0}z^{3} \div xyz^{-2}$,
13. $\dfrac{S^{-2}-r^{-2}}{S+r}$,
14. $a^{-3}(a+a^{2})$,
15. $(r^{-1}+u^{-1})(r^{-1}+2u^{-1})$,
16. $(x^{2}y^{\frac{1}{2}}-2)(x^{2}y^{\frac{1}{2}}+3)$,
17. $(R^{\frac{1}{3}}-S^{\frac{1}{3}})(R^{\frac{2}{3}}+R^{\frac{1}{3}}S^{\frac{1}{3}}+S^{\frac{2}{3}})$,
18. $(x^{3}-y^{-2})^{2}$.

Compute:

19. $(1.06)^{4} \cdot (1.06)^{-3}$,
20. $(\frac{2}{3})^{-1}(\frac{3}{2})^{-2}$,
21. $(-8)^{\frac{1}{3}} - 4^{-\frac{1}{2}} + 1$,
22. $100 \cdot \dfrac{1}{(1.03)^{3}} \cdot \dfrac{1}{(1.03)^{-2}} \cdot (1.03)^{4}$,
23. $\dfrac{(-32)^{\frac{1}{5}}(8)^{\frac{1}{3}}(225)^{-\frac{1}{2}}}{(100)^{-1} \cdot (15)^{-1}}$,
24. $4^{\frac{1}{2}} \cdot 4^{0} \cdot 4^{2} - 31$,
25. $(1.05)^{-2} - \dfrac{1}{1.05}$,
26. $1.02 - (1.02)^{-1}$
27. $(\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3})$,
28. $(27^{-\frac{1}{3}} - \frac{2}{3})[\frac{2}{3} - (\frac{1}{3})^{0}]$.

72. Operations with Exponents and Radicals. In many problems of business and finance, operations involving exponents and radicals occur in the extraction of roots, simplification of exponential expressions, and solving certain equations involving powers and roots. Tables giving powers and approximate roots of numbers are available in most handbooks and aid in making computations. Certain powers and roots for numbers from 1 to 100 are given on pages 377 to 379. We shall see later that logarithms are useful in dealing with exponential expressions and in solving exponential equations. At times, however, we shall need to be thoroughly familiar with the rules for exponents in making computations without the aid of tables and other labor-saving devices, for it is almost impossible to rely entirely upon such devices without an understanding of the processes and operations involved. The following types of simple computations should be thoroughly mastered.

Illustrations:

1. What is the value of 23 cubed?

In Table II, page 378, go down the left-hand column to the number 23. Opposite 23, in the column marked "Cube," we find 12,167. Hence

$$23^3 = 12{,}167.$$

2. What is the value of 5.7 squared?

In the table, go down the left-hand column to the number 57. Opposite 57, in the column marked "Square," we find 3,249. Since there must be two decimal places in the product, we write,

$$5.7^2 = 32.49.$$

3. What is the value of $\sqrt{75} + \sqrt{48} - \sqrt{27}$?

We may first simplify the expression by writing

$$\sqrt{25 \times 3} + \sqrt{16 \times 3} - \sqrt{9 \times 3} = 5\sqrt{3} + 4\sqrt{3} - 3\sqrt{3}$$
$$= 6\sqrt{3}.$$

In the table, we find $\sqrt{3} = 1.7321$. Hence, $6\sqrt{3} = 6 \times 1.7321 = 10.3926$.

4. Find $(\frac{3}{5})^3$.

Since the quantity within parentheses is to be used as a factor three times, we write

$$\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} = \frac{27}{125}. \qquad \text{Hence,} \qquad \left(\frac{3}{5}\right)^3 = \frac{3^3}{5^3} = \frac{27}{125}.$$

5. Find $\sqrt{\frac{3}{7}}$.

We may avoid the use of two approximate numbers from the table if we first *rationalize* the denominator by multiplying both the numerator and the denominator of the fraction by $\sqrt{7}$. We have

$$\sqrt{\frac{3}{7}} = \frac{\sqrt{3}}{\sqrt{7}} = \frac{\sqrt{3}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{21}}{7}.$$

In the table we find $\sqrt{21} = 4.5826$. Hence, $\sqrt{\frac{3}{7}} = 4.5826 \div 7 = 0.65$.

6. Write $\sqrt{200x^3}$ in simplest radical form.

$$\sqrt{200x^3} = \sqrt{100 \times x^2 \times 2x} = 10x \times \sqrt{2x}.$$

7. Write $\sqrt[4]{9x^2y^4}$ in simplest radical form.

$$\sqrt[4]{9x^2y^4} = y\,\sqrt[4]{(3x)^2} = y(3x)^{\frac{2}{4}} = y(3x)^{\frac{1}{2}} = y\,\sqrt{3x}.$$

Thus, we see that if the exponent of the radicand and the index have a common factor, it may be divided out.

Problems

By making use of the table on pages 377 to 379, wherever possible, find the value of each of the following:

1. 19^2,
2. 3^8,
3. 12^4,
4. $(\frac{4}{5})^3$,
5. 93^3,
6. $(0.06)^2$,
7. $(3.7)^3$,
8. $\frac{4^3}{7^3}$,
9. $(0.05)^5$,
10. $8^3 + 6^2$,
11. $5^2 \times 3^2 \times 4^3$,
12. $\sqrt{144} + \sqrt[3]{729}$,
13. $\sqrt{529}$,
14. $\sqrt{6{,}241}$,
15. $\sqrt{\frac{43}{85}}$,
16. $\sqrt{98} + \sqrt{288} + \sqrt{50}$,
17. $37^3 \times 1.5^2$,
18. $\sqrt[3]{250} - \sqrt[3]{128}$,
19. $\sqrt{\frac{17}{29}}$,
20. $\sqrt{\frac{93}{82}}$,
21. $2\sqrt{48} + \sqrt{12} - \sqrt{27}$,
22. $\sqrt[3]{81} - \frac{5}{2}\sqrt[3]{24}$,
23. $3\sqrt{45} + 2\sqrt{125} - \sqrt{500}$,
24. $\sqrt[3]{32} - \sqrt{12}$.

Express in simplest radical form:

25. $\sqrt{8}$,
26. $\sqrt{1{,}000}$,
27. $\sqrt[3]{24}$,
28. $\sqrt{48x^3}$,
29. $\sqrt{98i^3r^4}$,
30. $\sqrt[3]{54R^4S^5}$,
31. $\sqrt[3]{128S^7t^2}$,
32. $\sqrt[4]{162x^{10}y^3}$,
33. $\sqrt{0.75p^7qs^{10}}$,
34. $\sqrt{800(1+i)^3}$,
35. $\sqrt[4]{36}$,
36. $\sqrt[4]{25}$,
37. $\sqrt[4]{100x^2y^2}$,
38. $\sqrt[4]{4i^4j^8}$,
39. $\sqrt[6]{27z^{18}y^3}$,
40. $\sqrt[6]{8u^3v^9}$,
41. $\sqrt[4]{10{,}000S^2T^6}$,
42. $\sqrt[4]{\dfrac{i^6}{49}}$,
43. $\sqrt[6]{0.008q^{15}}$,
44. $\sqrt{242x^3} \times \sqrt[4]{64x^6}$,
45. $\sqrt[3]{(1.05)^4} \times \sqrt[6]{(1.05)^4P^7}$.

A knowledge of exponents is quite useful in solving many types of equations encountered in financial and business problems. A more extensive treatment will be given in a later chapter, but a few illustrations and problems at this time will show their application in some of the simpler equations and formulas.

Illustrations:

1. Solve the equation $N^{\frac{2}{3}} = 36$.

The instructions embodied in the exponent of N are to find the number such that when we take its cube root and then square the result we shall have 36. To find N, we make use of the fact that

$$N = (N^{\frac{2}{3}})^{\frac{3}{2}},$$

which follows from the law for raising a power to a power. Hence, by raising both sides of the given equation to the $\frac{3}{2}$ power, we see that

$$N = 36^{\frac{3}{2}} = 6^3 = 216.$$

2. Find the value of the expression $S(1+i)^{-n}$, when $S = 200$, $i = 0.04$, and $n = 2$.

Making the substitutions, we have

$$200(1.04)^{-2} = \frac{200}{(1.04)^2} = \frac{200}{1.0816} = 184.91.$$

PROBLEMS

1. Solve the following for x:

(*a*) $x^{\frac{1}{3}} = 3$,
(*b*) $x^{\frac{2}{3}} = 16$,
(*c*) $x^{-\frac{1}{3}} = 2$,
(*d*) $\frac{1}{2}x^{-\frac{1}{2}} = 5$,
(*e*) $2x^{\frac{3}{2}} = +54$,
(*f*) $\frac{2}{3}x^{\frac{3}{4}} = 18$,
(*g*) $(1 + x)^{\frac{1}{3}} = 1.03$,
(*h*) $384 = 3 \times 2^x$,
(*i*) $\frac{1}{32} = 4(\frac{1}{2})^x$.

2. Simplify and find the value of the following expressions:

(*a*) $P(1 + r)^n$, when $P = 500$, $r = 0.08$, and $n = 2$;

(*b*) $R\left[\dfrac{1 - (1 + i)^{-n}}{i}\right]$, when $R = 100$, $i = 0.06$, and $n = 2$;

(*c*) $R\left[\dfrac{(1 + i)^n - 1}{i}\right]$, when $R = 200$, $i = 0.03$, and $n = 2$;

(*d*) $A(1 + r)^{-n}$, when $A = 1{,}000$, $r = 0.03$, and $n = 2$;

(*e*) $\dfrac{a - ar^n}{1 - r}$, when $a = 3$, $r = 2$, and $n = 6$;

(*f*) ar^{n-1}, when $a = -\frac{1}{2}$, $r = -\frac{1}{2}$, $n = 6$;

(*g*) $\dfrac{ar^n - a}{r - 1}$, when $r = -1$, $a = 2.3$, $n = 26$;

(*h*) $R\left[\dfrac{(1 + i)^{-n} - 1}{-i}\right]$, when $R = 50$, $i = 0.04$, $n = 1$;

(*i*) $\dfrac{i}{1 - (1 + i)^{-n}}$, when $i = 0.025$, $n = 1$;

(*j*) $p[(1 + i)^{1/p} - 1]$, when $p = 2$, $i = 0.21$;

(*k*) $\dfrac{i}{p(1 + i)^{1/p} - 1}$, when $p = 2$, $i = 0.44$;

(*l*) $P\left(1 + \dfrac{j}{m}\right)^{mn}$, when $P = 1{,}000$, $j = 0.06$, $m = 2$, $n = 1$.

73. Square Root. When neither tables of square roots nor computing machines are available, it is often necessary to find a square root by arithmetic. The process may be explained through algebra but is perhaps best demonstrated by means of examples. In the first illustration below, each step is shown in the order in which the operations are carried out.

Illustration 1: Find the square root of 591.285.

1. Point off the number into groups of two places each, beginning at the decimal point and working both ways. The root will have one digit for each group, and the decimal point will be directly above the decimal point in the number.

$$\begin{array}{l} \qquad\;\; . \\ \sqrt{5|91.28|50} \end{array}$$

2. Find the largest square in the first, or left-hand, group, and enter its square root as the first digit in the answer. At the same time, place this square below the first group, and subtract.

$$\begin{array}{l} \;\;2\;\;\;. \\ \sqrt{5|91.28|50} \\ \;\;\underline{4} \\ \;\;1 \end{array}$$

3. Bring down the next group for a partial remainder.

$$\begin{array}{l} \;\;2 \\ \sqrt{5|91.28|50} \\ \;\;\underline{4\;\;\;\;\;} \\ \;\;1\;91 \end{array}$$

4. Double the root already found for a trial divisor.

$$2 \times 2 = 4$$

5. Place the trial divisor to the left of the partial remainder, leaving enough space to write another digit in the completed divisor.

$$\begin{array}{rl} & \;\;2\;\;\;. \\ & \sqrt{5|91.28|50} \\ & \;\;\underline{4\;\;\;\;\;} \\ 4 & \;\;1\;91 \end{array}$$

6. For the moment, disregard the right-hand digit in the partial remainder, and divide the rest by the trial divisor for a trial quotient.

$$19 \div 4 = 4+$$

7. Write the trial quotient as the last digit in the completed divisor, and also as a new digit in the answer.

$$\begin{array}{rl} & \;\;2\;\;4. \\ & \sqrt{5|91.28|50} \\ & \;\;\underline{4\;\;\;\;\;} \\ 44 & \;\;1\;91 \end{array}$$

8. Multiply the completed divisor by the trial quotient, and subtract the product from the remainder.

$$\begin{array}{rl} & \;\;2\;\;4. \\ & \sqrt{5|91.28|50} \\ & \;\;\underline{4\;\;\;\;\;} \\ 44 & \;\;1\;91 \\ \underline{4} & \\ 176 & \;\;\underline{1\;76} \\ & \;\;\;\;\;15 \end{array}$$

9. Repeat steps 4 to 8 until the root is found to as many decimal places as desired. The work necessary to carry out the answer to two decimal places is given below.

$$\begin{array}{rrl}
 & 2\ \ 4.\ 3\ \ 1 & \\
 & \sqrt{5|91.28|50} & \\
 & 4 & \\
44 & \overline{1\ 91} & \text{(Trial divisor 4)} \\
 & 1\ 76 & \\
483 & \overline{15\ 28} & \text{(Trial divisor 48)} \\
 & 14\ 49 & \\
4861 & \overline{79\ 50} & \text{(Trial divisor 486)} \\
 & 48\ 61 & \\
 & \overline{30\ 89} &
\end{array}$$

Sometimes, the product found in step 8 turns out to be larger than the partial remainder from which it is to be subtracted. When this happens, we must decrease the trial quotient by one point and retrace steps 7 and 8. A little practice will enable the worker to foresee this situation and use the correct quotient in the first place.

Illustration 2: Find the square root of 3,961.

$$\begin{array}{rrl}
 & 6\ \ 3. & \\
 & \sqrt{39|61} & \\
 & 36 & \\
123 & \overline{3\ 61} & \text{(Trial divisor 12)} \\
 & \underline{3\ 69} &
\end{array}$$

We see that the trial quotient 3 is too large; so we must change it to 2. By making a mental trial we might have anticipated this situation and written 2 in the first place instead of trying to use 3. We have

$$\begin{array}{rrl}
 & 6\ \ 2.\ 9 & \\
 & \sqrt{39|61.00} & \\
 & 36 & \\
122 & \overline{3\ 61} & \text{(Trial divisor 12)} \\
 & 2\ 44 & \\
1249 & \overline{1\ 17\ 00} & \text{(Trial divisor 124)} \\
 & \underline{1\ 12\ 41} &
\end{array}$$

When the trial divisor in step 6 turns out to be larger than that part of the partial remainder into which it is to be divided, a zero must be entered in the trial divisor and in the answer, and another group brought down. The work then continues as before.

Illustration 3: Find the square root of 921.86.

$$\begin{array}{rrl}
 & 3\ \ 0.\ 3\ \ 6 & \\
 & \sqrt{9|21.86|00} & \\
 & \underline{9} & \\
603 & 21\ 86 & \text{(Trial divisor 60)} \\
 & \underline{18\ 09} & \\
6066 & 3\ 77\ 00 & \text{(Trial divisor 606)} \\
 & \underline{3\ 63\ 96} &
\end{array}$$

We see that 6 was too large to divide into 2 after the 21 was brought down. We therefore place a zero beside the 6, making the trial divisor 60, and also place a zero in the answer. Then the 86 is brought down, making the partial remainder now 2,186. The work is then carried out as shown in steps 6 to 8.

PROBLEMS

Mark off the following numbers into groups of two places each, and place the decimal point in the correct position; then extract the square root to three decimal places.

1. 355.50.
2. 876.
3. 397.6.
4. 888.431.
5. 1.083.
6. 1.06.
7. 0.065.
8. 0.2.
9. 0.00368.
10. 731.86424.

Obtain the square root of the following numbers to two decimal places, and round off to one decimal place; hence obtain the square root *correct to one decimal place:*

11. 934.
12. 6.
13. 2.
14. 89.34.
15. 3.905.
16. 1,056.7.

74. Formula Graphs. The compound amount A of a principal P for a period of 5 years at 6% is expressed by the formula

$$A = P(1.06)^5.$$

Taking $(1.06)^5 = 1.34$, this formula may be written $A = 1.34P$.

This mathematical statement may be represented graphically by computing a set of values satisfying the formula and drawing a line graph of the data secured. It is usually desirable to draw a smooth curve through the plotted points, especially when a great many points have been used. The calculations for values of P equal to \$100, \$200, \$300, and so forth, to \$800 are given below, and the corresponding graph is shown in Figure 3.

VALUE OF P · 1.34 = VALUE OF A

P	A
\$100	\$ 134
200	268
300	402
400	536
500	670
600	804
700	938
800	1.072

The graph of the formula $A = P(1.06)^5$ is a visual representation of the relationship between a given principal and the resulting amount determined by this principal through the algebraic statement. Since the interest rate (0.06) and the time (5) are *constant*, there are but two *variables* in the formula. Since the amount A depends upon whatever principal P is selected, we say that A is the *dependent variable* and P is the *independent variable* and that A is a *function* of P.

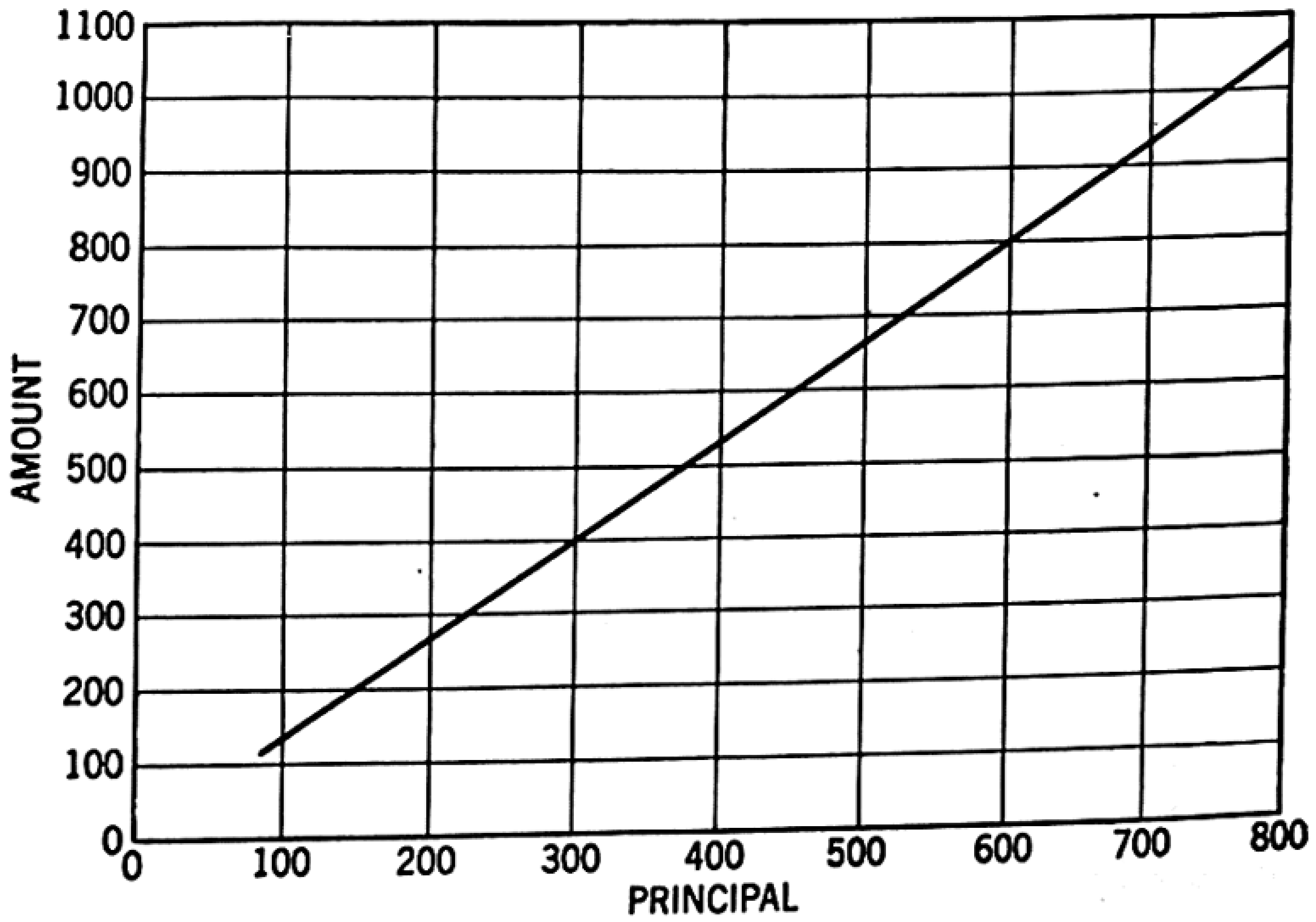


Figure 3. Graph of $A = P(1.06)^5$

In general, it is said that one variable is a function of another variable if a relationship exists between the two such that when the second is given, the first is determined. This relationship may be expressed in words, in a table of values, in an algebraic formula, or in a graph, as we have done in Figure 3.

In making a graph to show the relationship between an independent variable and a dependent variable, it is conventional to locate pairs of corresponding values with respect to a pair of perpendicular lines, called the *coordinate axes*. One of these reference lines is drawn horizontal. Then negative values of the independent variable are laid off to the left of the vertical line, and negative values of the dependent variable are laid off below the horizontal line.

Illustrations:

1. Graph the equation $y = 2x - 4$.

As written, the equation would indicate that x is the independent variable, y the dependent variable. Values of x are selected arbitrarily, and the corresponding y values are found by substitution into the equation. A few of these pairs of values are shown in Figure 4.

x	y
0	−4
1	−2
2	0
3	2
−1	−6
−2	−8

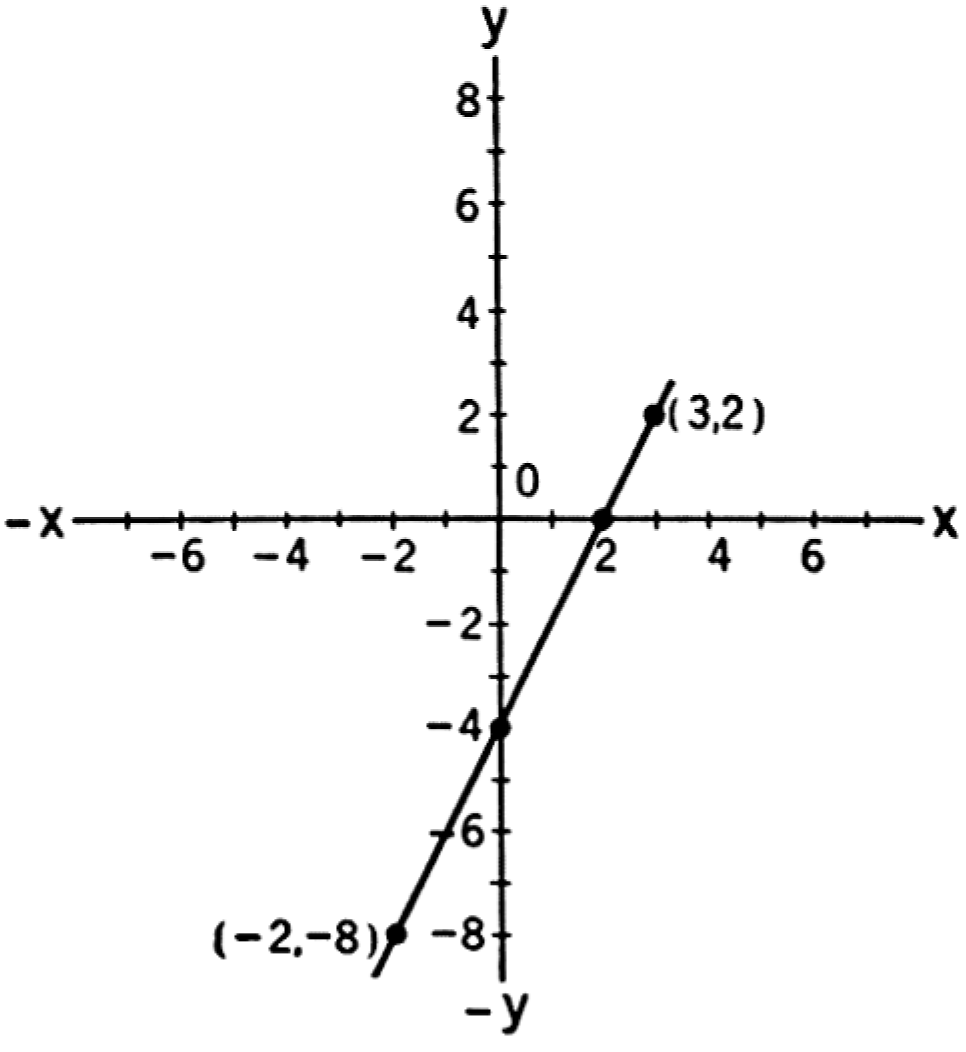


Figure 4. Graph of $y = 2x - 4$

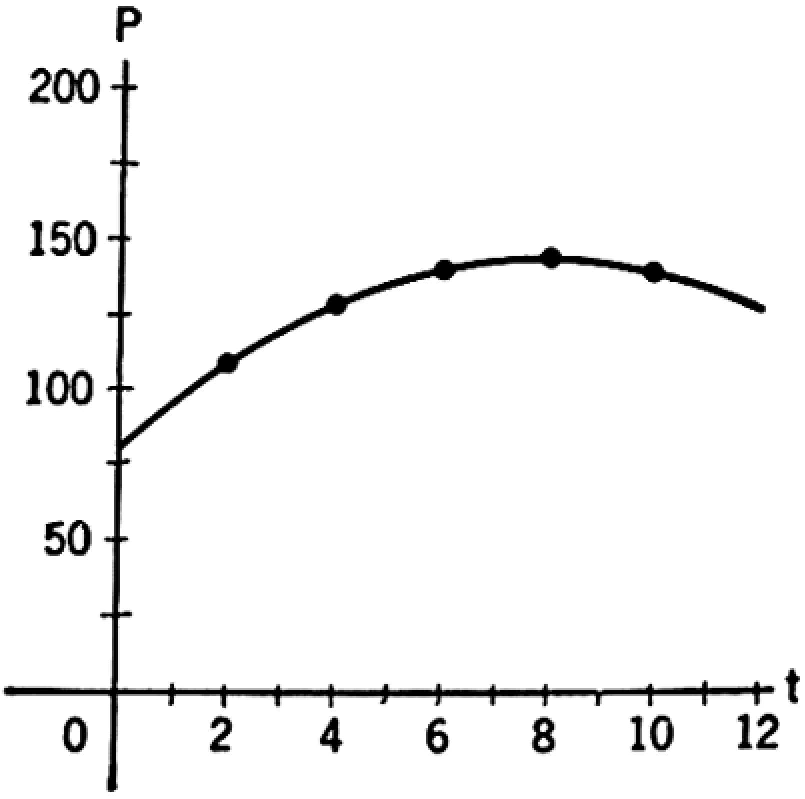
The pairs of values $x = 0$, $y = -4$; $x = 1$, $y = -2$; and so forth, are usually simply written as $(0, -4)$, $(1, -2)$, and so forth, and are referred to as the *coordinates* of the respective points on the graph. The point (3,2), for example, is located by moving 3 units to the right of the y axis and 2 units up from the x axis. To plot the point $(-2, -8)$, we move 2 units to the left of the y axis and 8 units below the x axis. When sufficiently many points have been plotted to indicate the nature of the relationship, a smooth curve (in this case, a straight line) is sketched through the points to complete the graph as shown in Figure 4.

2. An economist in studying price changes developed the following formula relating the price of a certain commodity to the time elapsed since the end of World War II.

$$P = 80 + 16t - t^2,$$

in which t is the time in months and P is the price.

To observe the graph of the formula for a period of 1 year, we first find values of P corresponding to $t = 0, 1, 2, 3$, and so forth, to 12, and then plot the "points" thus obtained on coordinate paper, using perpendicular axes for t and P (see Figure 5).



t	P
0	80
1	95
2	108
3	119
4	128
5	135
6	140
7	143
8	144
9	143
10	140
11	135
12	128

Figure 5. Graph of $P = 80 + 16t - t^2$

A smooth curve drawn through the plotted points completes the graph of the formula. The curve (Figure 5) indicates that the price of the commodity rose steadily for the first 8 months and then began to fall. The peak of the curve is sometimes referred to as a *maximum,* as contrasted with a low point in a dip, which is called a *minimum*, although, of course, more precise definitions would be required in a thorough analysis.

PROBLEMS

Draw graphs of the following formulas or equations:

1. $L = 2.4d$. (Take at least 5 negative and 5 positive values of d.)
2. $y = 2x + 1$. (Take integral values of x from -3 to $+4$.)
3. $S = (1.06)^n$. (Take $n = 1, 2, 3$, and 4.)

In Problems 4 to 19 take convenient values for the independent variable, and sketch a smooth curve through the plotted points.

4. $y = 3x - 2$.
5. $y + x = 4$.
6. $2y + 4x = 3$.
7. $x = 2y - 3$.
8. $y = 2x^2$.
9. $y = \sqrt{x}$.
10. $2y^2 = x$.
11. $y = \sqrt{16 - x^2}$.
12. $y = \sqrt{25 + x^2}$.
13. $I = P(0.04)(3)$.
14. $y = 3^x$.
15. $y = 3^{-x}$.
16. $y = 2 \cdot 3^x$.
17. $y = 3^x + 4$.
18. $A = \pi r^2$.
19. $C = D^2 + 5D + 6$.

Exercises

1. Add the following expressions:

(*a*) $L_x + 2L_{x+1} - 3L_{x+2}$ and $4L_x - 6L_{x+1} + 9L_{x+2}$,
(*b*) $C - RS_k$, $4C + 5RS_k$, and $3RS_k$,
(*c*) $an + nL$, $2an - 3nL$, and $an + nL + 15$,
(*d*) $ar^4 + ar^3 + ar^2 + ar + a$, and $-ar - ar^2 - ar^3$,
(*e*) $4xy - 3x^2y + 2xy^2$, and $xy^2 + x^2y$.

2. Simplify:

(*a*) $5x - 4y - (2x - 3y)$,
(*b*) $(1 + 2r + r^2) - (r - r^2)$,
(*c*) $x_1y_1 + x_2y_2 + x_3y_3 - \dfrac{x_1y_1 + x_2y_2 + x_3y_3}{3}$,
(*d*) $(6n - 7n) - (2n + 3n) - (-3n + 5n)$,
(*e*) $\left(\dfrac{j}{m} + 1\right) - \left(\dfrac{j}{m}\right)$.

Find the following products, and simplify your answers:

3. $(3x + 2)(3x - 2)$.

4. $(7x - 6)(4x + 3)$.

5. $(R - S)(R^2 + RS + S^2)$.

6. $-\frac{7}{x}\left(-3x + \frac{2xy}{7}\right)$.

7. $(P + 2)(2P - 40)(P - 1)$.

Factor the following expressions:

8. $32m^2 - 50n^2$.

9. $28a^2 - 8a - 20$.

10. $3R^2 + 43R - 30$.

11. $a^2b^2c^2 - a^2b^2d^2$.

12. $\frac{1}{2}x^2 + \frac{3}{2}x + 1$.

13. Divide $2x^2 - 11x - 21$ by $2x + 3$. (*Suggestion:* Factor and then simplify.)

14. Divide the sum of $(a + b)$ and $(a - b)$ by the product of a and b.

15. Obtain by inspection the following products:

(*a*) $3x(4x - 1)$,
(*b*) $(2x + 1)(2x - 1)$,
(*c*) $(r - 5)(r + 4)$,
(*d*) $2p^2(4p - 3q)$,
(*e*) $3s(3s + 2)(3s - 2)$.

16. Factor the following:

(*a*) $P + Pni$,
(*b*) $9x^2 - 25y^2$,
(*c*) $d^2 - d - 6$,
(*d*) $k^2 + 5k - 24$
(*e*) $3m^3 - 6m^2 - 24m$
(*f*) $P(1 - d_1)(1 - d_2) - P(1 - d_1)(1 - d_2)d_3$,
(*g*) $C(1 - r) - (1 - r)Cr$,
(*h*) $\frac{n(n - 1)}{2}i^2 + \frac{n(n - 1)(n - 2)}{6}i^3$.

17. Simplify by factoring and cancellation:

(*a*) $\frac{4x^2 - 9y^2}{6y - 4x}$,
(*b*) $\frac{t^2 + 4t + 4}{t + 2}$,
(*c*) $\frac{(P + C)K}{2K + 4KP}$,
(*d*) $\frac{6x^2 - 7x - 3}{4x^2 - 12x + 9}$,
(*e*) $\frac{a^2 + 2ab + b^2}{a - b} \cdot \frac{a^2 - 2ab + b^2}{a + b}$.

18. Expand:

(*a*) $(x + y)^4$, (*c*) $(1 + R)^3$, (*e*) $(2x - 1)^4$,
(*b*) $(x - y)^3$, (*d*) $(2r + q)^3$, (*f*) $[a + (1 + r)]^2$.

19. Perform the indicated operations:

(*a*) $\dfrac{x-3}{2x} + \dfrac{3x-7}{2x}$, (*d*) $\dfrac{2}{3} + \dfrac{a}{a+b} + \dfrac{b}{a-b}$,

(*b*) $\dfrac{a-c}{b-d} - \dfrac{a}{b}$, (*e*) $\dfrac{1}{1-x} + \dfrac{x}{x^2-1} + \dfrac{x-1}{(x-1)^2}$.

(*c*) $\dfrac{1-r}{r^4} + \dfrac{1}{r^2}$,

Graph the formulas of Problems 20 to 28 by drawing a smooth curve through at least five sets of values.

20. $I = 2{,}400(0.06)t$.
21. $I = (1.03)^n$.
22. $P = \dfrac{100}{(1.06)^n}$.
23. $y = 3x + 4$.
24. $A = 4D + 2D^2$.
25. $S = e^{2t}$. (Take $e = 2.7$.)
26. $3y = 6x - 2$.
27. $S = \$100(1.06)^n$.
28. $x^2 = y + 4$.

29. Simplify:

(*a*) $x^2y^2(xy^2)^2$,
(*b*) $(abc)(a^2b^2c)(abc)^2$,
(*c*) $(2a^0)(2a)^0$,
(*d*) $x^{n-1} \cdot x^{1-n}$ $(x \neq 0)$,
(*e*) $(-4x^2y^3)(6xy^2)$,
(*f*) $(3x^{-2}y^5)(-2x^3y^{-4})$,
(*g*) $6^n \cdot 6^n$,
(*h*) $5a^{\frac{2}{3}}b^{-\frac{1}{2}} \cdot 3a^{\frac{1}{2}}b^{\frac{1}{3}}$,
(*i*) $c^{-\frac{1}{3}}(2d)^{-\frac{1}{3}}(cd)$,
(*j*) $2^n \div 2^3$,
(*k*) $8a^{-4}b^{-2} \div 4ab$,
(*l*) $a^2\left(\dfrac{1}{4a}\right)^2$.

30. Simplify:

(*a*) $7^2 \times 10^8$,
(*b*) $\sqrt[4]{16c^4d^8}$,
(*c*) $\sqrt{\dfrac{144}{11}}$,
(*d*) $\dfrac{1}{3}\sqrt{\dfrac{9}{y^2}}$,
(*e*) $2\sqrt{2} \times 7\sqrt{3}$,
(*f*) $(2\sqrt{27} - \sqrt{48})^2$.

31. Find the square root of each of the following numbers correct (rounded off) to two decimal places:

(*a*) 75.90, (*d*) 0.0144, (*g*) 202.04,
(*b*) 4,780, (*e*) 0.1440, (*h*) 27.035,
(*c*) 1,026.34, (*f*) 0.00367 (*i*) 9.0064.

Self-Test

Time: 50 Minutes

(Score 5 points for each problem correctly solved)

1. Subtract the sum of $m + 2n$ and $n - 2m$ from $4m - 4n$.

2. If $L_x + L_{x+1}$ persons each receive $D + 2$ dollars in dividends at the end of each year for N years, what is the total amount of dividends received?

3. Multiply $(1 - a)(1 - b)(1 - c)$.

4. What is the sum of $(a + L) + (a + L) + (a + L)$ and so forth, for n terms?

5. Simplify:

(*a*) $(2x^{-1}y^2)(-3x^2y^{-2})$, (*c*) $(a^{-2}b^{\frac{1}{2}})^2$

(*b*) $2a^{\frac{1}{2}}b^{\frac{2}{3}} \cdot a^0b^{-\frac{1}{3}}$, (*d*) $4a^{-2}b^{-1} \div 2ab$.

6. Simplify:

(*a*) $\sqrt{27} - \sqrt{48}$, (*b*) $\sqrt{24}/\sqrt{6}$, (*c*) $\sqrt[4]{81x^4y^8}$.

7. What is the square of $1 + \frac{j}{m}$?

8. Write in simplified form:

$$R\frac{N_{x+1}}{D_x} \div \frac{3N_{x+1}}{2D_x}.$$

9. Divide $\frac{Prt}{100 + k}$ by $\frac{P}{100}$.

10. Perform the indicated operations:

$$\frac{\frac{P + S}{R} - 1}{R}.$$

11. Write in simplest form the product of $\frac{27 - x^3}{27x^3}$ and $\frac{9x^2}{3 - x}$.

12. What is the LCM of $m + n$, $m - n$, $m^2 - n^2$, and m?

13. Divide $2x^2 + x - 10$ by $x - 2$.

14. Write in expanded form: $(H + T)^3$.

15. What is the square of $x + y + r$?

16. The graph of the formula $L = 16 + d$ passes through what point where $d = 7$?

17. Between what two values of t does the graph of the equation $S = 4 + 6t - t^2$ pass through a "maximum" point?

18. What is the value of y where the graph of the equation $2y = 3x - 7$ crosses the y axis?

19. Draw the graph of $A = P(1 + i)^n$, $P = 1$, $i = 0.05$, using $n = 1, 2, 3, 4$, and so forth. Describe the shape of the curve. Does A increase or decrease with n?

20. Find the square root of 2,360 correct to two decimal places.

Chapter VII

LOGARITHMS

Introduction. Many calculations that occur in business and industrial problems entail lengthy arithmetic operations. Even the simple task of multiplying \$437.63 by 92.64 and dividing the product by 352.7 turns out to be a rather lengthy calculation when carried out in the usual manner by the use of multiplication and division. Workers vary immensely in the rapidity and accuracy with which they perform arithmetic computations; but no degree of competence eliminates the vast amount of time and labor required when no labor-saving device is employed.

Computations involved in various problems can often be carried out in a much more efficient manner by the use of logarithms—a labor-saving device invented early in the seventeenth century. By means of logarithms, multiplication and division of numbers are reduced to the simpler operations of addition and subtraction of their corresponding logarithms, and such operations as the extracting of a root and the raising to a power are reduced to the simpler operations of multiplication and division. Calculating machines, tables, and slide rules are all valuable aids to computation, but logarithms have a place in the business world which cannot be overlooked by the student of business and finance.

The present chapter deals with the meaning and use of logarithms and is intended to give the student a working knowledge of this labor-saving device. One of the many important applications of logarithms will be found in the solutions of equations arising in connection with such problems as compound interest and payment of debts, which will be discussed in the next chapter.

75. The Meaning of Logarithms. In the study of exponents certain numbers were expressed in terms of powers of 10. For example,

$$
\begin{aligned}
1 &= 10^0,\\
10 &= 10^1,\\
100 &= 10^2,\\
1{,}000 &= 10^3,\\
10{,}000 &= 10^4,\\
100{,}000 &= 10^5,
\end{aligned}
$$

and so forth.

But what about the numbers between 1 and 10, between 10 and 100, between 100 and 1,000, and so forth? Does it not seem reasonable that if the proper exponent were known in each case, every positive number might be expressed in terms of 10 raised to a power? Furthermore, would we not expect that if the number is between 1 and 10, the exponent needed would lie between 0 and 1? Is it not evident that, for numbers between 10 and 100, the needed exponent would lie between 1 and 2; for numbers between 100 and 1,000, the exponent would lie between 2 and 3, and so forth? In fact, it *is* true that every positive number can be expressed, to any desired degree of approximation, in terms of 10 raised to a power. For example:

$1 = 10^{0.00000}$,
$2 = 10^{0.30103}$,
$6 = 10^{0.77815}$,
$8 = 10^{0.90309}$,
$9 = 10^{0.95424}$,
$10 = 10^{1.00000}$,

$26 = 10^{1.41497}$,
$54 = 10^{1.73239}$,
$82 = 10^{1.91381}$,
$97 = 10^{1.98677}$,
$100 = 10^{2.00000}$,
$248 = 10^{2.39445}$,
$462 = 10^{2.66464}$,

$875 = 10^{2.94201}$,
$990 = 10^{2.99564}$,
$1{,}000 = 10^{3.00000}$,
$1{,}964 = 10^{3.29314}$,
$3{,}472 = 10^{3.54058}$,
$7{,}965 = 10^{3.90119}$,
$9{,}240 = 10^{3.96567}$.

The exponents above are correct to five decimal places, and the statements are of the form

$$N = 10^x, \tag{1}$$

in which to every positive number N there corresponds a definite value x. (It can be seen that a negative number does not have a real number for its exponent when the base is positive. For example, when x is negative, 10^{-x} may be written $1/10^{x}$, which is positive.) The exponent x is called the *logarithm* of N to the *base* 10. Hence, *a logarithm is an exponent—a special exponent that corresponds to a number which may be expressed as a base raised to a power.* Bases other than 10 are often used in connection with certain phases of mathematics, but the base 10 is most convenient for most computations.

According to the definition of logarithms we say that the logarithm of 1 to the base 10 equals 0, the logarithm of 2 to the base 10 equals 0.30103, the logarithm of 6 to the base 10 equals 0.77815, and so forth. In general, when referring to a number N whose logarithm is x, we may say that the logarithm of N to the base 10 equals x. In abbreviated algebraic notation, this is written

$$\log_{10} N = x, \tag{2}$$

or simply $\log N = x$, where it is understood that the base is 10. Equations (1) and (2) give the same information in two different ways. The first is called the *exponential* form; the second, the *logarithmic* form.

In logarithmic form, the preceding statements may, therefore, be written as follows:

$$\begin{array}{rcl@{\qquad}rcl}
\log 1 &=& 0.00000, & \log 100 &=& 2.00000,\\
\log 2 &=& 0.30103, & \log 248 &=& 2.39445,\\
\log 6 &=& 0.77815, & \log 462 &=& 2.66464,\\
\log 8 &=& 0.90309, & \log 875 &=& 2.94201,\\
\log 9 &=& 0.95424, & \log 990 &=& 2.99564,\\
\log 10 &=& 1.00000, & \log 1{,}000 &=& 3.00000,\\
\log 26 &=& 1.41497, & \log 1{,}964 &=& 3.29314,\\
\log 54 &=& 1.73239, & \log 3{,}472 &=& 3.54058,\\
\log 82 &=& 1.91381, & \log 7{,}965 &=& 3.90119,\\
\log 97 &=& 1.98677, & \log 9{,}240 &=& 3.96567.
\end{array}$$

PROBLEMS

1. Write the following statements in logarithmic form:

(*a*) $10{,}000 = 10^4$,
(*b*) $2{,}550 = 10^{3.40654}$,
(*c*) $87.55 = 10^{1.94226}$,
(*d*) $6.25 = 10^{0.79588}$,
(*e*) $35 = 10^{1.54407}$,
(*f*) $3.5 = 10^{0.54407}$,
(*g*) $8 = 10^{0.90309}$,
(*h*) $120 = 10^{2.07918}$,
(*i*) $8{,}422 = 10^{3.92542}$,
(*j*) $842.2 = 10^{2.92542}$,
(*k*) $84.22 = 10^{1.92542}$,
(*l*) $8.422 = 10^{0.92542}$,
(*m*) $13 = 10^{1.11394}$,
(*n*) $1{,}000{,}000 = 10^6$,
(*o*) $101 = 10^{2.00432}$,
(*p*) $75 = 10^{1.87506}$.

2. Write the following statements in exponential form:

(*a*) $\log 5 = 0.69897$,
(*b*) $\log 45 = 1.65321$,
(*c*) $\log 290 = 2.46240$,
(*d*) $\log 5{,}325 = 3.72632$,
(*e*) $\log 5.25 = 0.72016$,
(*f*) $\log 45.9 = 1.66181$,
(*g*) $\log 290.95 = 2.46382$,
(*h*) $\log 5{,}325.2 = 3.72634$,
(*i*) $\log 2{,}345 = 3.37014$,
(*j*) $\log 234.5 = 2.37014$,
(*k*) $\log 23.45 = 1.37014$,
(*l*) $\log 2.345 = 0.37014$.

3. Fill in the blanks in the following table:

	Number	*Base*	*log*
(*a*)	32	2	______
(*b*)	______	4	3
(*c*)	9	______	2
(*d*)	49	______	2
(*e*)	216	6	______
(*f*)	$\frac{1}{4}$	2	______
(*g*)	0.001	______	−3
(*h*)	$\frac{1}{8}$	______	3
(*i*)	______	4	−2
(*j*)	______	6.342	0

76. Characteristic and Mantissa. We have seen that a number between 1 and 10 has a logarithm between 0 and 1, written 0 plus some decimal. If the number is between 10 and 100, the logarithm is 1 plus some decimal. A number between 100 and 1,000 has a logarithm equal to 2 plus some decimal. In general, the logarithm of a number will consist of an integral part and a decimal part. The integral part is called the *characteristic* and the positive decimal part, the *mantissa*. It can be shown, for example, that the logarithm of 5,278 = 3.72247 to five decimal places. In this case, the characteristic is 3, and the mantissa is 0.72247.

Consider, now, positive numbers less than 1. From the definition of logarithms it is evident that the two forms of statements in the following table are equivalent:

Exponential Form	*Logarithmic Form*
$10^0 = 1.$,	$\log 1 = 0$,
$10^{-1} = 0.1$,	$\log 0.1 = -1$,
$10^{-2} = 0.01$,	$\log 0.01 = -2$,
$10^{-3} = 0.001$,	$\log 0.001 = -3$,
$10^{-4} = 0.0001$,	$\log 0.0001 = -4$,
and so forth.	and so forth.

It appears from the foregoing expressions that the logarithm of a number between 0 and 1 is negative. It also appears that the logarithm of a number between 1 and 0.1 lies between 0 and -1 and hence equals -1 plus some decimal. Similarly, it is indicated that the logarithm of a number between 0.1 and 0.01 lies between -1 and -2 and hence equals -2 plus some decimal, and so forth. To illustrate,

$$\log 0.43 = -1 + \text{the mantissa } 0.63347,$$

$$\log 0.08 = -2 + \text{the mantissa } 0.90309,$$

$$\log 0.004 = -3 + \text{the mantissa } 0.60206,$$

$$\log 0.00075 = -4 + \text{the mantissa } 0.87506.$$

The mantissa, or the decimal part of a logarithm, may be found in a table of logarithms. It would be more correct to call such a table a table of mantissas, since the characteristic, or the integral part, is not given in the tables and must be determined by inspection. The characteristic is found by simply observing whether the number lies

Between 1 and 10,
Between 10 and 100,
Between 100 and 1,000,
Between 1,000 and 10,000,
and so forth,

or

Between 1 and 0.1,
Between 0.1 and 0.01,
Between 0.01 and 0.001,
Between 0.001 and 0.0001,
and so forth.

This discussion leads naturally to the following rules for determining the characteristic of the logarithm of a number.

Rule 1. *The characteristic for a number greater than* 1 *is a positive integer* (*or zero*) *which is one less than the number of digits in the given number to the left of the decimal point.*

For example, since

7.6 lies between 1 and 10 (10^0 and 10^1), $\log 7.6 = 0. +$ the mantissa,
89 lies between 10 and 100 (10^1 and 10^2), $\log 89 = 1. +$ the mantissa,
358.215 lies between 100 and 1,000 (10^2 and 10^3), $\log 358.215 = 2. +$ the mantissa,
5,408.2 lies between 1,000 and 10,000 (10^3 and 10^4), $\log 5408.2 = 3. +$ the mantissa.

Problems

What is the characteristic of the logarithm of each of the following numbers?

1. 5,
2. 3.36,
3. 53,
4. 253,
5. 3.29,
6. 3.297,
7. 356.2,
8. 2358,
9. 350,000,
10. 16.12,
11. 1.001,
12. 1,001.01
13. 6.0007,
14. 3,262,000,
15. 2,
16. 1,002.7.

Rule 2. *The characteristic for a number between* 0 *and* 1 *is a negative integer which is one more than the number of zeros between the decimal point and the first figure other than zero.*

For example, since

0.6 lies between 1 and 0.1 (10^0 and 10^{-1}), $\log 0.6 = -1. +$ the mantissa, 0.07 lies between 0.1 and 0.01 (10^{-1} and 10^{-2}), $\log 0.07 = -2. +$ the mantissa,

0.003 lies between 0.01 and 0.001 (10^{-2} and 10^{-3}), $\log 0.003 = -3. +$ the mantissa,

0.00024 lies between 0.001 and 0.0001 (10^{-3} and 10^{-4}), $\log 0.00024 = -4. +$ the mantissa.

It must be kept in mind that although the characteristic for a number less than 1 is negative, the mantissa is positive. Thus, $\log 0.0525 = -2 + 0.72016$. The negative portion of the logarithm and the positive portion could, of course, be combined, and we could write $\log 0.0525 = -1.27984$. But, in doing this, the decimal part of the logarithm no longer corresponds to the proper number sequence in the table of logarithms.

To avoid this confusion, one of two procedures is usually followed: Sometimes the logarithm of a number less than 1 is written by placing the negative sign of the characteristic above the characteristic—namely, $\log 0.0525 = \bar{2}.72016$. Another method, which many persons prefer, is to add 10 to the characteristic and subtract 10 at the right of the mantissa. By this latter method we would write $\log 0.0525 = 8.72016 - 10$. Thus, upon obtaining the necessary mantissas from the logarithm table, we would write

$$\begin{aligned}
\log 0.6 &= -1 + 0.77815 = \bar{1}.77815 = 9.77815 - 10,\\
\log 0.07 &= -2 + 0.84510 = \bar{2}.84510 = 8.84510 - 10,\\
\log 0.003 &= -3 + 0.47712 = \bar{3}.47712 = 7.47712 - 10\\
\log 0.00024 &= -4 + 0.38021 = \bar{4}.38021 = 6.38021 - 10.
\end{aligned}$$

For the method of adding and subtracting 10, the rule for determining the characteristic of the logarithm of a number less than 1 can now be stated:

The characteristic for a number between 0 *and* 1 *is found by subtracting the number of zeros immediately to the right of the decimal point from* 9, *and then placing* -10 *after the mantissa.*

Thus, if there are no zeros immediately to the right of the decimal point, we write 9.mantissa -10; one zero, 8.mantissa -10; two zeros, 7.mantissa -10; and so forth. This method will be the one used for computations in this book.

Problems

In two ways, state the characteristic of the logarithm of each of the following numbers:

1. 0.1,
2. 0.3,
3. 0.82,
4. 0.974,
5. 0.01,
6. 0.03,
7. 0.087,
8. 0.076,
9. 0.001,
10. 0.003,
11. 0.0029,
12. 0.0045,
13. 0.0003,
14. 0.00032,
15. 0.0001,
16. 0.00069,
17. 0.00001,
18. 0.000074,
19. 0.00002,
20. 0.00000045.

77. Finding the Logarithm of a Number. On page 358 there begins a "five-place" table of logarithms. In the left-hand column of this table under *N* are found the first three digits of the number whose logarithmic mantissa is to be found in the body of the table. The fourth digit of the number is found in the top row of the table, indicating the column in which the mantissa is to be found. The mantissas are approximate numbers that have been computed by mathematical means to five significant figures. For greater accuracy in computations made by logarithms, tables having more significant figures are available in many handbooks. A portion of Table I appears as follows:

N		0	1	2	3	4	5	6	7	8	9
600	77	815	822	830	837	844	851	859	866	873	880
01		887	895	902	909	916	924	931	938	945	952
02		960	967	974	981	988	996	*003	*010	*017	*025
03	78	032	039	046	053	061	068	075	082	089	097

To find the logarithm of 6,003, we locate 600 in the column under *N* and move across the page to the column headed 3, and find there 77837 for the mantissa. Applying the rule for the characteristic and inserting the decimal point, we write log 6003 = 3.77837.

To find the logarithm of 0.06027, we locate 602 under *N* and move across the page to the column headed 7, and find 78010 for the mantissa. We use 78010 instead of 77010 because mantissas, beginning with the one marked with an asterisk (*), start with the same digits as the next row. Employing the rule for writing the characteristic, we write log 0.06027 = 8.78010 − 10.

PROBLEMS

1. Refer to the portion of the logarithm table given on page 217 and verify that:

(*a*) $\log 6.01 = 0.77887$,
(*b*) $\log 6.008 = 0.77873$,
(*c*) $\log 6.022 = 0.77974$,
(*d*) $\log 60.25 = 1.77996$,
(*e*) $\log 60.14 = 1.77916$,
(*f*) $\log 60.1 = 1.77887$,
(*g*) $\log 602.8 = 2.78017$,
(*h*) $\log 600 = 2.77815$,
(*i*) $\log 6{,}016 = 3.77931$,
(*j*) $\log 6{,}029{,}000 = 6.78025$,
(*k*) $\log 0.06001 = 8.77822 - 10$,
(*l*) $\log 0.602 = 9.77960 - 10$,
(*m*) $\log 0.0006 = 6.77815 - 10$,
(*n*) $\log 0.006026 = 7.78003 - 10$.

2. Find the logarithm of each of the following:

(*a*) 600.7,
(*b*) 60.15,
(*c*) 6,000,
(*d*) 60.38,
(*e*) 6,039,
(*f*) 0.06003,
(*g*) 0.00601,
(*h*) 0.0006028,
(*i*) 0.00006038,
(*j*) 0.06.

3. Refer to the table of logarithms beginning on page 357 and find the logarithm of:

(*a*) 892,
(*b*) 506,
(*c*) 37.7,
(*d*) 4.28,
(*e*) 0.05619,
(*f*) 0.004,
(*g*) 3,450,
(*h*) 0.000739,
(*i*) 9.95,
(*j*) 63.72,
(*k*) 103.2,
(*l*) 0.06667,
(*m*) 13.07,
(*n*) 1,025,000,
(*o*) 0.7298.

78. Finding the Number Corresponding to a Logarithm. The number corresponding to a given logarithm is called the *antilogarithm*. Thus, 6.01 is the antilogarithm of 0.77887; 600 is the antilogarithm of 2.77815. To find the number whose logarithm is known and tabulated, we look in the table and find the mantissa, noting the number at the top of the column in which it is found, as well as the sequence opposite in the left-hand column. We then write down the entire sequence of digits corresponding to the given mantissa and place the decimal point according to the rules for characteristics.

Illustrations:

1. Find the number whose logarithm is 1.77909.

In the table, the mantissa 0.77909 is in the 3 column, opposite 601. Hence, the given mantissa corresponds to the sequence 6013, and the rule for characteristics places the decimal point between the 0 and the 1, giving 60.13 as the number to be found. (Since the characteristic is one less than the number of digits to the left of the decimal, it follows that the decimal point must be placed so that the number of digits to its left exceeds the characteristic by one.)

2. Find the number whose logarithm is 7.77996 − 10.

In the table, we find that the mantissa appears in the 5 column, opposite 602. We write down the sequence 6025. The rule for characteristics tells us that there must be two zeros immediately to the right of the decimal point. Therefore, the number whose logarithm is 7.77996 − 10 is 0.006025.

Problems

Find the numbers whose logarithms are:

1. 1.77866,
2. 2.78025,
3. 3.77895,
4. 9.77873 − 10,
5. 8.77924 − 10,
6. 7.77960 − 10,
7. 1.55630,
8. 2.94699,
9. 3.34143,
10. 0.59704,
11. 7.73472 − 10,
12. 8.95289 − 10,
13. 5.67006,
14. 6.50934 − 10,
15. 0.16077.

79. Interpolation. Table I (page 357) gives the five-place mantissa of the logarithm of any four-digit number. As we have seen, we can read directly the mantissa of any number of four digits or less from this table. If, however, the number contains more than four digits, we are unable to obtain its logarithm directly from the table but we may obtain an approximate value by a process of *interpolation*. It is assumed in this process that small changes in the number are directly proportional to small changes in the logarithm. The approximation is obtained simply by adding the proportional part of the difference between two successive mantissas to the smaller one. Interpolation may also be used to find a number whose logarithm is given but whose mantissa is not listed in the table. The following examples will illustrate the process in each case.

Illustration 1: Find the logarithm of 60.057.

The characteristic is 1. With the aid of the table, we find

$$\begin{aligned} \log 60.050 &= 1.77851 \qquad \text{(mantissa difference} = 8, \\ \log 60.060 &= 1.77859. \qquad \text{disregarding decimal),} \end{aligned}$$

The number 60.057 lies 0.7 of the way from 60.050 to 60.060. By proportion, therefore, the mantissa of the logarithm of 60.057 is taken 0.7 of the way from 77851 to 77859. Hence, we add $0.7 \times 8 = 5.6$, or approximately 6, to 77851 to get

$$\log 60.057 = 1.77857.$$

The interpolation may be arranged as follows:

$$\begin{array}{ccccc}
 & \textit{Digits} & \qquad & & \textit{Mantissa} \\
10 & \left\{\begin{array}{l}\left.\begin{array}{l}60050\\ 60057\end{array}\right\}\ 7\\ 60060\end{array}\right. & \qquad & x & \left.\begin{array}{r}\left\{\begin{array}{c}77851\\ M\end{array}\right.\\ 77859\end{array}\right\}\ 8
\end{array}$$

In this arrangement it is easy to see that the value of x in the proportion

$$7:10 = x:8,$$

added to 77851, will give the mantissa M that is to be found. Solving the equation, we have

$$\begin{aligned} 10x &= 56, \\ x &= 5.6, \text{ or 6 (approximately).} \end{aligned}$$

Hence, as found above, the mantissa M equals $77851 + 6 = 77857$, and $\log 60.057 = 1.77857$.

PROBLEMS

1. Given $\log 6{,}005 = 3.77851$ and $\log 6{,}006 = 3.77859$, find the following:

(*a*) log 60,052,
(*b*) log 60,055,
(*c*) log 60,058,
(*d*) log 600.52,
(*e*) log 60.057,
(*f*) log 0.60059.

2. Given log 0.4756 = 9.67724 − 10 and log 0.4757 = 9.67733 −10, find the following:

(*a*) log 0.47565,
(*b*) log 4.7562,
(*c*) log 0.0047568,
(*d*) log 475,650,
(*e*) log 47,560,
(*f*) log 475.66.

Illustration 2: Find the number whose logarithm is 8.77977 − 10.

The number will have one zero to the right of the decimal. From the table we find

log 0.060220 = 8.77974 − 10 (number difference = 10,
log 0.060230 = 8.77981 − 10. disregarding decimal),

The sequence 77977 lies $\frac{3}{7}$ of the way from 77974 to 77981. By proportion, therefore, the number we seek will lie $\frac{3}{7}$ of the way from 0.060220 to 0.060230. Hence, we add $\frac{3}{7} \times 10 = \frac{30}{7}$, or approximately 4, to the sequence 6022 to get 60224. Hence,

$$8.77977 - 10 = \log 0.060224.$$

The interpolation may be arranged as follows:

$$\begin{array}{cc} \textit{Digits} & \textit{Mantissa} \\ 10\left\{\begin{array}{l}\left.\begin{array}{c}60220\\ N\end{array}\right\}\ x\\ 60230\end{array}\right. & \left.\begin{array}{r}3\left\{\begin{array}{c}77974\\ 77977\end{array}\right.\\ 77981\end{array}\right\}7 \end{array}$$

The proportion may be expressed

$$3:7 = x:10.$$

Solving for x, we have

$$7x = 30,$$
$$x = 4\tfrac{2}{7}, \text{ or 4 (approximately).}$$

Hence, as found above, the number sequence N equals 60220 + 4 = 60224, and

0.060224 is the number whose logarithm is 8.77977 − 10.

It will be observed that the decimal point must be placed in accordance with the appropriate rule for characteristics.

PROBLEMS

1. The mantissa corresponding to the number 98.14 is 0.99185. What mantissa corresponds to the number 9.814? 0.09814? 0.009814?

2. The mantissas corresponding to the numbers 2,024 and 2,025 are 0.30621 and 0.30643, respectively. (*a*) What is the mantissa difference? (*b*) What number sequence would have a mantissa of 0.30632? 0.30625? 0.30636? 0.30639?

3. What numbers have the following logarithms: (*a*) 3.30632, (*b*) 1.30625, (*c*) 9.30636 − 10, (*d*) 7.30639?

4. With the aid of the logarithm table find the numbers whose logarithms are (*a*) 1.38077, (*b*) 2.46260, (*c*) 9.76016 − 10, (*d*) 6.91410 − 10.

80. Using Logarithms in Computations. We have discussed the meaning of logarithms and have seen how they are related to exponents. This was necessary in order to understand the principles underlying operations with logarithms in computations. Now that we have also learned to find the number corresponding to a given logarithm, we are in a position to consider how logarithms may be used in problems involving multiplication, division, raising to a power, and extracting roots. The general laws pertaining to each of these operations will be suggested through examples.

(1) **Multiplication.** Find the product of 2.350 and 34.80 by the use of logarithms.

Since every positive number may be written in terms of 10 raised to some exponent, we may find in the table of logarithms the proper exponents for the two given numbers and write

$$2.350 = 10^{0.37107},$$

and

$$34.80 = 10^{1.54158}.$$

The product of the two numbers may now be written as

$$2.350 \times 34.80 = 10^{0.37107} \times 10^{1.54158} = 10^{1.91265}.$$

In the study of exponents we learned that the product of the two exponential forms above may be expressed as a power of the common base 10 by *adding the exponents*, and we know from our study of the previous sections that *the exponent* 1.91265 *is a* *logarithm.* Furthermore, it is the logarithm of the product we are calculating. We may, therefore, turn to the table of logarithms and find that number which corresponds to 1.91265. The number found will be the product of the two given numbers and by the law for characteristics will have two places to the left of the decimal point. The number turns out to be 81.78. By actual multiplication, the student may easily verify the answer.*

The process illustrated by this example can easily be generalized. In symbols, if A, B, C, D, and so forth, represent positive numbers,

$$\log (ABCD \cdots) = \log A + \log B + \log C + \log D + \cdots .$$

That is,

Theorem I. *The logarithm of a product equals the sum of the logarithms of its factors.*

Illustrations:

1. Find the product of 75.03 × 34.9 × 0.035 by using logarithms.

From the table we find the logarithm of each factor and list them in a column:

$$\begin{aligned} \log 75.03 &= 1.87523 \\ \log 34.9 &= 1.54283 \\ \log 0.035 &= \underline{8.54407 - 10} \\ &\ 11.96213 - 10 \end{aligned}$$

The logarithms are then added, and we obtain 11.96213 − 10, or, more simply, 1.96213. The *antilogarithm* is then determined by obtaining from the table the number sequence corresponding to the mantissa 0.96213 and placing the decimal point so as to have two digits to its left. The answer turns out to be 91.65.

2. Find the value of (462)(−56).

Since a negative number does not have a real number for its logarithm when the base is positive, we avoid the difficulty by noting that the effect of the negative sign is to give a negative product and compute 462 × 56, writing the minus sign before the product. We have

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* It is to be observed that computations made with the use of a table of logarithms will usually be approximate rather than exact. The accuracy will depend upon the number of "places" given in the tables as well as the number of digits in the numbers entering into the computations. For the purpose of learning basic procedures used in business, statistics, finance, and related fields, a five-place table is quite satisfactory. However, one should have available more extensive tables for actual work in the field.

$$\begin{array}{rcl} \log 462 & = & 2.66464 \\ \log \ \ 56 & = & \underline{1.74819} \\ \log (462)(56) & = & 4.41283 \end{array}$$

Hence, $(462)(56) = 25{,}872,$

and $(462)(-56) = -25{,}872.$

3. Add the following logarithms:

$$(a)\ \begin{array}{r} 2.68954 \\ \underline{1.34212} \\ 4.03166 \end{array} \qquad (c)\ \begin{array}{r} 9.87257 - 10 \\ \underline{9.93262 - 10} \\ 19.80519 - 20, \text{ or } 9.80519 - 10 \end{array}$$

$$(b)\ \begin{array}{l} \ \ 3.42971 \\ \ \ \underline{9.82535 - 10} \\ 13.25506 - 10, \text{ or } 3.25506 \end{array}$$

PROBLEMS

1. We have learned that when we multiply numbers we *add* their logarithms. Add the following "logs," and write the results in simplified form. What are the characteristics obtained? What will be the nature of the corresponding numbers?

$$(a)\ \begin{array}{r} 3.52157 \\ \underline{2.10724} \end{array} \qquad (d)\ \begin{array}{r} 9.95636 - 10 \\ \underline{9.97425 - 10} \end{array}$$

$$(b)\ \begin{array}{l} 3.56821 \\ \underline{6.46264 - 10} \end{array} \qquad (e)\ \begin{array}{l} 9.99652 - 10 \\ \underline{5.41367 \qquad\ \ } \end{array}$$

$$(c)\ \begin{array}{r} 8.34682 - 10 \\ \underline{9.26788 - 10} \end{array}$$

2. Find the value of the following by using logarithms:

(*a*) $(88)(-92)$, (*c*) $(-99.56)(0.072)$,
(*b*) $(265)(-906)(-7.8)$, (*d*) $(-172)(28)(0.937)$.

3. Find the following products by using logarithms, and then verify your answers by actual multiplication. What is the amount of the error in each case?

(*a*) 12 × 144,
(*b*) 250 × 950,
(*c*) 4,968 × 7.26,
(*d*) 225 × 45.12,
(*e*) 53.89 × 27.819,
(*f*) 452.71 × 0.574,
(*g*) 25,002 × 67.80.

4. Find the following products by using logarithms:

(*a*) 54.86 × 2.4637,
(*b*) 64.49 × 89.064 × 3.06 × 4.41,
(*c*) 0.895 × 0.574 × 0.05842,
(*d*) 3,728,600 × 0.007281,
(*e*) $1.9 \times 10^{10} \times 0.06673$,
(*f*) $2.4 \times 10^{6} \times 3{,}923.5$,
(*g*) 1,220 × 1.6238 × 2.437 × 25.6,
(*h*) 1,000 × 1.05 × 1.0007,
(*i*) 67,724 × 18.923 × 600.14,
(*j*) 0.0003216 × 0.0089235 × 17.

(2) **Division.** Divide 264.82 by 4.163 by using logarithms. The two numbers may be written in terms of powers of 10 by making use of the definition of a logarithm. We have

$$\log 264.82 = 2.42295,$$
$$\log 4.163 = 0.61941.$$

In exponential form we may write

$$264.82 = 10^{2.42295},$$
$$4.163 = 10^{0.61941}.$$

The quotient of the two numbers may now be expressed

$$264.82 \div 4.163 = 10^{2.42295} \div 10^{0.61941}.$$

In our study of exponents we learned that $10^a \div 10^b = 10^{a-b}$. Applying this principle of exponents, we may write

$$264.82 \div 4.163 = 10^{2.42295-0.61941} = 10^{1.80354}.$$

The exponent 1.80354 is seen to be that one which when used as a power of 10 will yield the desired quotient and is, therefore, the logarithm of that quotient. The antilogarithm of 1.80354 is, therefore, the quotient of the two given numbers. The quotient found in this manner turns out to be 63.612.

In dividing the two numbers above, we subtracted their logarithms and then found the antilogarithm of this difference. The proof of the law for a quotient where M and N are any two positive numbers is as follows:

Let $M = 10^x$ and $N = 10^y$, in which case $x = \log M$, $y = \log N$.

Then
$$\frac{M}{N} = 10^x \div 10^y = 10^{x-y}.$$

Hence, $\log \frac{M}{N} = x - y = \log M - \log N.$

That is,

Theorem II. *The logarithm of a quotient is equal to the logarithm of the numerator minus the logarithm of the denominator.*

Illustrations:

1. Divide 327.24 by 12.71 by using logarithms.

From the table we find the logarithm of each number and write

$$\begin{aligned} \log 327.24 &= 2.51487 \\ \log 12.71 &= \underline{1.10415} \\ &\ 1.41072 \end{aligned}$$

The logarithms are then subtracted, and we obtain 1.41072. For the mantissa 0.41072, we find in the table the number sequence 25746, and since the characteristic is 1, we place the decimal point between the 5 and the 7. Hence, the quotient is 25.746.

2. Divide 950 by -65.

$$\begin{aligned} \log 950 &= 2.97772 \\ \log\ \ 65 &= \underline{1.81291} \\ \log(950 \div 65) &= 1.16481 \end{aligned}$$

Hence,
$$950 \div 65 = 14.615,$$

and
$$(950) \div (-65) = -14.615.$$

3. Subtract the following logarithms:

$$(a)\ \begin{array}{l} 8.72358 - 10 \\ \underline{2.53973} \\ 6.18385 - 10 \end{array} \quad (b)\ \begin{array}{l} 3.58287 \\ \underline{9.34212 - 10} \end{array} \quad \text{Write as } \begin{array}{r} 13.58287 - 10 \\ \underline{9.34212 - 10} \\ 4.24075 \end{array}$$

(c) $$\begin{array}{r} 8.37256 - 10 \\ \underline{9.47161 - 10} \end{array} \qquad \text{Write as } \begin{array}{r} 18.37256 - 20 \\ \underline{9.47161 - 10} \\ 8.90095 - 10 \end{array}$$

PROBLEMS

1. We learned that when we divide two numbers we subtract their logarithms. Subtract the following logarithms, and write the results in simplified form. What are the characteristics obtained? What will be the nature of the corresponding numbers?

(a) $$\begin{array}{r} 3.72841 \\ \underline{1.60428} \end{array}$$ (b) $$\begin{array}{r} 2.72151 \\ \underline{6.92142 - 10} \end{array}$$ (c) $$\begin{array}{r} 8.23144 - 10 \\ \underline{9.72162 - 10} \end{array}$$

(d) $$\begin{array}{r} 7.84323 - 10 \\ \underline{1.04725 } \end{array}$$ (e) $$\begin{array}{r} 0.84327 \\ \underline{3.32168} \end{array}$$ (f) $$\begin{array}{r} 0.62540 \\ \underline{8.32102 - 10} \end{array}$$

2. Divide the following by using logarithms, and then check by actual division: (a) 3,751 ÷ 250, (b) 14.682 ÷ 2.061, (c) 246.83 ÷ 23.75, (d) 0.00231 ÷ 6,004.

3. Find the quotient for each of the following by using logarithms:

(a) $24.637 \div 2.86$, (f) $(-128) \div 4$,
(b) $476.7 \div 26.205$, (g) $(-13.76) \div (-2.973)$,
(c) $0.8371 \div .92449$, (h) $273.65 \div (-0.634)$,
(d) $6.589 \div 0.83007$, (i) $56{,}280 \div (-938{,}600)$.
(e) $0.024 \div 75{,}627$,

4. The following problems involve both multiplication and division. Find the value of each by making use of the two theorems just illustrated.

(a) $\dfrac{27.73 \times 8.57}{-6.3784}$, (d) $\dfrac{1{,}000}{0.000713} \times (-14.634)$,

(b) $\dfrac{0.8785 \times 3.1045}{250 \times 1.06}$, (e) $\dfrac{6.3 \times 10^{11}}{121.37} \times 0.0003004$,

(c) $\dfrac{430 \times 89.2 \times 1.0409}{24.8 \times 52.9 \times 0.95552}$, (f) $16 \times \dfrac{0.14321}{967.8}$,

(g) $\dfrac{1.4 \times 10^7}{6.3 \times 10^5} \times \dfrac{1}{18}$,

(h) $\dfrac{0.014329}{66.31} \times \dfrac{217.8}{3{\cdot}2677} \times 100{,}320{,}000$,

(i) $\dfrac{1.19 \times 10^{-7}}{0.02365}$, (j) $\dfrac{(1.02)(1.035)}{1.04} \times 5{,}000$.

(3) **Raising a Number to a Power.** Find the value of $(1.06)^3$. Since $(1.06)^3 = 1.06 \times 1.06 \times 1.06$, we may write

$$\log (1.06)^3 = \log 1.06 + \log 1.06 + \log 1.06 \qquad \text{(from Theorem I)}$$

or

$$\log (1.06)^3 = 3 \log 1.06.$$

Hence, $\log (1.06)^3 = 3 \times 0.02531 = 0.07593.$

Thus, $(1.06)^3 = \text{antilog } 0.07593 = 1.1911.$

In finding the value of $(1.06)^3$, we multiplied the logarithm of 1.06 by the exponent 3 and then found the antilogarithm of the product. The proof of the law for raising any positive number N to the power x is as follows: Let $N = 10^y$, in which case $y = \log N$.

Then, $N^x = (10^y)^x = 10^{xy}.$

Hence, $\log N^x = xy = x \log N.$

That is,

Theorem III. *The logarithm of a number with an exponent is equal to the exponent times the logarithm of the number.*

Illustrations:

1. Find the value of $(8.25)^4$ by using logarithms.

From the table we find the logarithm of 8.25 and write

$$\log 8.25 = 0.91645,$$
$$\log (8.25)^4 = 4 \times 0.91645 = 3.66580,$$

Therefore, $(8.25)^4 = 4632.3$ (antilogarithm of 3.66580).

The logarithm of 8.25 is multiplied by the exponent 4, giving 3.66580. The mantissa 0.66580 when found in the table gives us the number sequence 46323 after proper interpolation, and the characteristic 3 tells us where to place the decimal point.

2. Find the value of $(-72)^3$.

The effect of the minus sign is to yield a negative answer.

$$\log 72 = 1.85733,$$

$$3 \log 72 = 5.57199,$$

Therefore, $$72^3 = 373{,}240,$$

and $$(-72)^3 = -373{,}240.$$

3. Multiply the following logarithms by the numbers indicated:

(a) $$\begin{array}{r} 3.71843 \\ 2 \\ \hline 7.43686 \end{array}$$

(b) $$\begin{array}{r} 8.31522 - 10 \\ 3 \\ \hline 24.94566 - 30 \end{array} \text{ or } 4.94566 - 10$$

PROBLEMS

1. We have learned that to raise a number to a power we multiply the logarithm of the number by the exponent and then find the antilogarithm of the product to obtain the answer desired. Multiply the following logarithms by the numbers indicated and find the antilogarithms from the table:

(a) $$\begin{array}{r} 3.24718 \\ 2 \\ \hline \end{array}$$

(c) $$\begin{array}{r} 9.92145 - 10 \\ 6 \\ \hline \end{array}$$

(e) $$\begin{array}{r} 6.01734 - 10 \\ 11 \\ \hline \end{array}$$

(b) $$\begin{array}{r} 8.57217 - 10 \\ 2 \\ \hline \end{array}$$

(d) $$\begin{array}{r} 2.74316 \\ 7 \\ \hline \end{array}$$

2. Compute the exact values of the following expressions by actual multiplication, and compare with the results obtained when logarithms are used: (a) 2^3, (b) $(-3)^3$, (c) $(1.2)^2$, (d) $(0.25)^2$; (e) $(-0.3)^4$, (f) $(400)^3$.

3. Find the value of each of the following by using logarithms: (a) $(1.04)^8$, (b) $(-0.07)^{10}$, (c) $(7.5)^5$, (d) $(1.03)^{20}$, (e) $(1.04)^{-6}$.

4. The following problems involve multiplication, division, and raising to a power. Find the value of each expression by using the three laws just learned.

(a) $\dfrac{3{,}507 \times (1.06)^5}{27.56}$,

(b) $\dfrac{36{,}059}{(1.03)^9}$,

(c) $\dfrac{-15{,}003}{(1.05)^{10}}$,

(d) $\dfrac{45.5 \times 3.1416 \times (6.3)^2}{36.92 \times 5.12}$,

(e) $\dfrac{900(1.04)^7}{(1.05)^4}$,

(f) $\dfrac{1{,}050.7}{(1.06)^3(1.05)^2}$,

(g) $(3.6)(1.7562)^{15}$,

(h) $5 \times (0.75)^{24}$,

(i) $\dfrac{600}{(1.035)^{17}}$,

(j) $1{,}000(1.0575)^{10}$,

(k) $2(\frac{1}{2})^{17}$,

(l) $7(\frac{1}{3})^{20}$,

(m) $\dfrac{100(1.02)^{25}}{(1.03)^{11}}$,

(n) $\dfrac{6.3072 \times (-14.67)^5}{0.0125}$,

(o) $\dfrac{999.99}{(1.732)^2(0.000738)}$.

(4) **Extracting a Root of a Number.** Find $\sqrt[3]{559.1}$.

We may write $\sqrt[3]{559.1} = (559.1)^{\frac{1}{3}}$, and since the number 559.1 may be expressed in terms of 10 raised to an exponent (the logarithm of 559.1), we may write

$$\sqrt[3]{559.1} = (10^{2.74749})^{\frac{1}{3}},$$

or

$$\sqrt[3]{559.1} = 10^{0.91583}.$$

The exponent 0.91583 is seen to be that one needed as a power of 10 to yield the desired root. The antilogarithm of 0.91583 will, therefore, give us the value of the root we seek. It turns out to be 8.2382. Hence,

$$\sqrt[3]{559.1} = 8.2382.$$

In finding the cube root of 559.1 we divided the logarithm of that number by 3 and then found the antilogarithm of the quotient. The proof of the law for extracting any root q of a positive number N is as follows: Let $N = 10^x$, in which case $x = \log N$.

Then, $$\sqrt[q]{N} = N^{1/q} = (10^x)^{1/q} = 10^{x/q}.$$

Hence, by the definition of a logarithm,

$$\log \sqrt[q]{N} = \frac{x}{q} = \frac{\log N}{q}.$$

That is,

Theorem IV. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

Illustrations:

1. Find $\sqrt{1{,}024}$.

$$\begin{aligned}\log 1{,}024 &= 3.01030,\\ \log \sqrt{1{,}024} &= \tfrac{1}{2}(3.01030) = 1.50515,\\ \sqrt{1{,}024} &= \text{antilog } 1.50515 = 32.\end{aligned}$$

2. Find $\sqrt[5]{0.83816}$.

$$\begin{aligned}\log 0.83816 &= 9.92333 - 10,\\ \log \sqrt[5]{0.83816} &= \tfrac{1}{5}(9.92333 - 10)\\ &= \tfrac{1}{5}(49.92333 - 50) = 9.98467 - 10,\\ \sqrt[5]{0.83816} &= \text{antilog } 9.98467 - 10 = 0.96532.\end{aligned}$$

3. Find $\sqrt[3]{-96.93}$.

$$\begin{aligned}\log 96.93 &= 1.98646,\\ \tfrac{1}{3}\log 96.93 &= 0.66215,\\ \sqrt[3]{96.98} &= \text{antilog } 0.66215 = 4.5936,\end{aligned}$$

and $$\sqrt[3]{-96.93} = -4.5936.$$

PROBLEMS

1. We have seen that extracting a root of a number by means of logarithms involves dividing the logarithm of the number by the index of the root. Divide the following logarithms by the indicated number, and state where the decimal point will be placed in the antilogarithm of the quotient:

(*a*) $\dfrac{2.67321}{3}$,

(*b*) $\dfrac{1.92031}{5}$,

(*c*) $\dfrac{0.24361}{4}$,

(*d*) $\dfrac{0.02369}{2}$,

(*e*) $\dfrac{9.22063 - 10}{2}$,

(*f*) $\dfrac{6.17355 - 10}{3}$,

(*g*) $\dfrac{8.00123 - 10}{5}$,

(*h*) $\dfrac{7.86773 - 10}{8}$,

(*i*) $\dfrac{9.01629 - 10}{3}$,

(*j*) $\dfrac{7.34341 - 10}{3}$.

2. Evaluate:

(*a*) $\sqrt[3]{9{,}301}$,

(*b*) $\sqrt[3]{-17.83}$,

(*c*) $\sqrt[3]{0.0927}$,

(*d*) $\sqrt[3]{672.8}$,

(*e*) $\sqrt[3]{1{,}045.6}$,

(*f*) $\sqrt[5]{-6.34}$,

(*g*) $\sqrt[4]{99.607}$,

(*h*) $\sqrt[3]{\dfrac{83.75}{0.072}}$,

(*i*) $6{,}700\sqrt[6]{6{,}700}$,

(*j*) $\dfrac{\sqrt{555.65}}{14.27}$,

(*k*) $\sqrt[3]{0.00732}\sqrt{69.302}$,

(*l*) $5\sqrt{92.6}$,

(*m*) $\sqrt[7]{0.8883}$,

(*n*) $\dfrac{\sqrt[3]{97.4}}{(2.01)^2}$,

(*o*) $\dfrac{(1.04)^6}{(1.05)^{\frac{1}{3}}}$.

3. The following problem involves all four of the laws just learned. An outline of the solution is given, and it is suggested that the student complete the problem by following the outline.

Find the value of $\dfrac{6{,}500 \times \sqrt{4.73}}{(1.03)^6}$.

$\log 6{,}500 = 3.81291,$

$\frac{1}{2}\log 4.73 =$ ——— $\qquad \log 4.73 = 0.67486,$

(add)

$6 \log 1.03 =$ ——— $\qquad \log 1.03 = 0.01284,$

(subtract)

$\log =$ ——— $\qquad$ *Ans.* = 11,839.

4. Make an outline for the solution, and find the value of the following:

(*a*) $\dfrac{45.84 \times 653 \times \sqrt[3]{50{,}376}}{330 \times 9.7 \times (0.07)^2}$,

(*b*) $\dfrac{551 \times 5\sqrt{56.84}}{(1.04)^{32}}$,

(*c*) $\sqrt{\dfrac{96.75}{82 \times 0.0042}}$.

81. Logarithmic Graphs. After instituting a radio advertising program an automobile dealer reported the following sales of passenger cars and trucks for a four-month period beginning Jan. 1:

AUTOMOBILE SALES

Month	*Passenger Cars*	*Trucks*
Jan.	120	8
Feb.	180	11
Mar.	270	16
Apr.	405	27

At a meeting of his salesmen the dealer wished to present the two sales records graphically in order that a comparison of the rates of increase for both passenger cars and trucks could be observed. What type of graph could be used for this purpose?

A line graph of the two sales records depicts the actual sales as shown in Fig. 6.

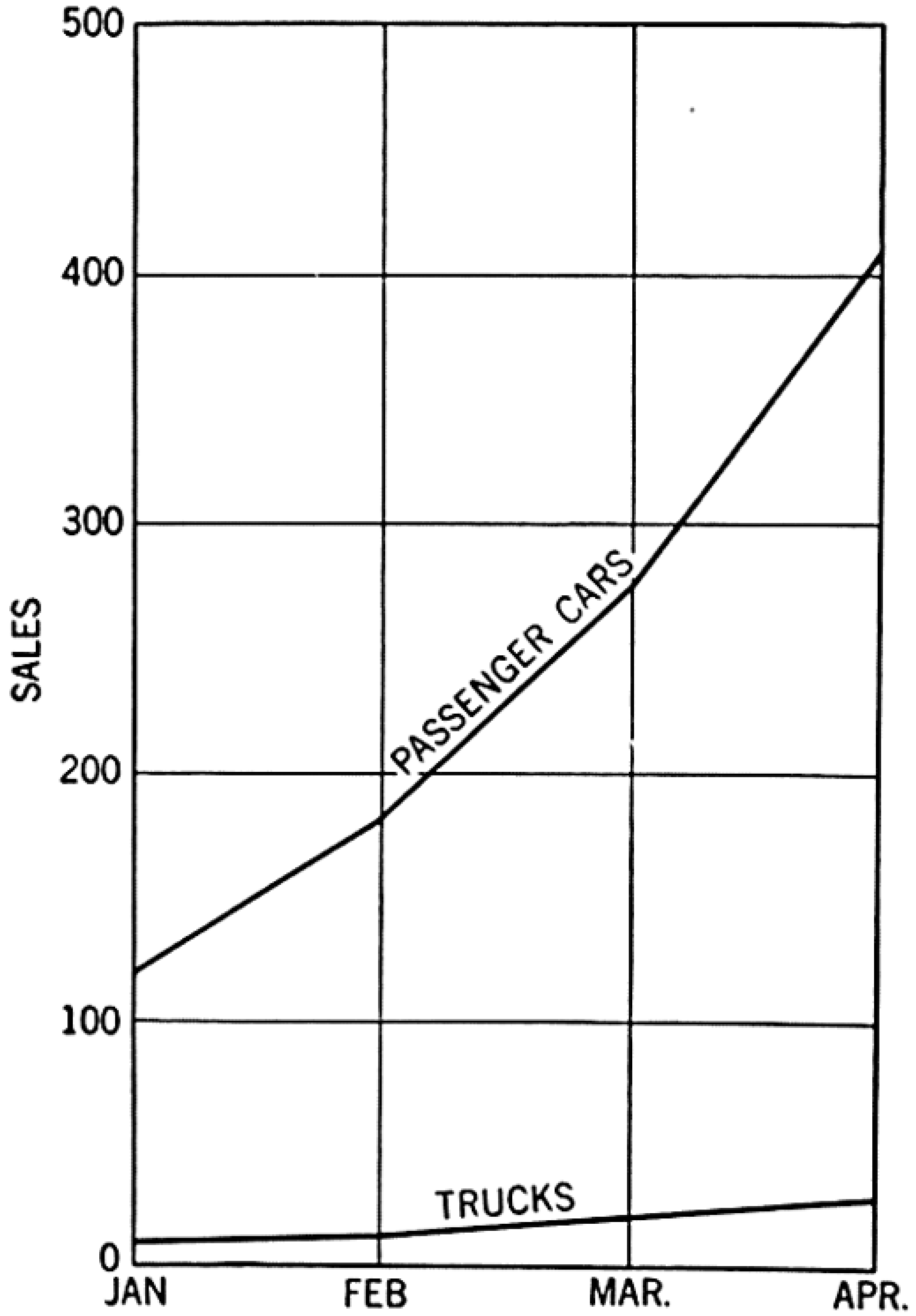

Figure 6. Sales records of passenger cars and trucks over a four-month period.

If the chief interest centered in the actual number of sales of each kind of vehicle, the line graph would serve well. However, it would be difficult to make a true comparison of the *rates of increase* from month to month on the basis of actual sales alone. We shall need some other type of graph that will show the true nature of the rate of increase for the two sales records.

The ratio of increase from month to month can be computed by dividing the number of vehicles sold in a given month by the number sold during the preceding month. Thus, the passenger sales for February were 180 ÷ 120, or $1\frac{1}{2}$ times those for January, whereas the truck sales ratio of increase for the same month was 11 ÷ 8, or $1\frac{3}{8}$. Since the divisions required to calculate the ratios can be performed by the use of logarithms, equal ratios or per cents of change may be represented by equal differences between the logarithms of the data. For example, suppose we wish to compare the rates of increase of sales of passenger cars as given above. The logarithm of 120 is 2.07918, and of 180 is 2.25527, giving a difference in logarithms of 0.17609. The logarithm of 270, the passenger sales for March, is 2.43136, giving a difference in "logs" between February and March sales of 0.17609. Since these two "log" differences are equal, we say that the rate of increase in sales in February over January is the same as the increase in March over February. An examination of the actual sales shows this to be the case. Therefore, by drawing a line graph for each set of logarithm values, we shall be able to compare the rates of sales increase for both sets of data. The logarithms of the numbers of sales are given in the accompanying table correct to 3 decimal places.

AUTOMOBILE SALES
(Logarithms of Numbers Sold)

Month	*Passenger Cars*	*Trucks*
Jan.	2.079	0.903
Feb.	2.255	1.041
Mar.	2.431	1.204
Apr.	2.607	1.431

A line graph of the logarithms of the number of sales of both passenger cars and trucks may be constructed on ordinary squared graph paper as shown on the opposite page.

Figure 7 shows that the sales of passenger cars and trucks increased at virtually constant and equal rates.

The necessity of determining the logarithms of numbers in making a graph of this kind may be obviated by using specially prepared paper known as *semilogarithmic paper*. This paper has the natural scale along the horizontal line and the logarithmic scale along the vertical line. The use of this type of graphing is illustrated in Fig. 8.

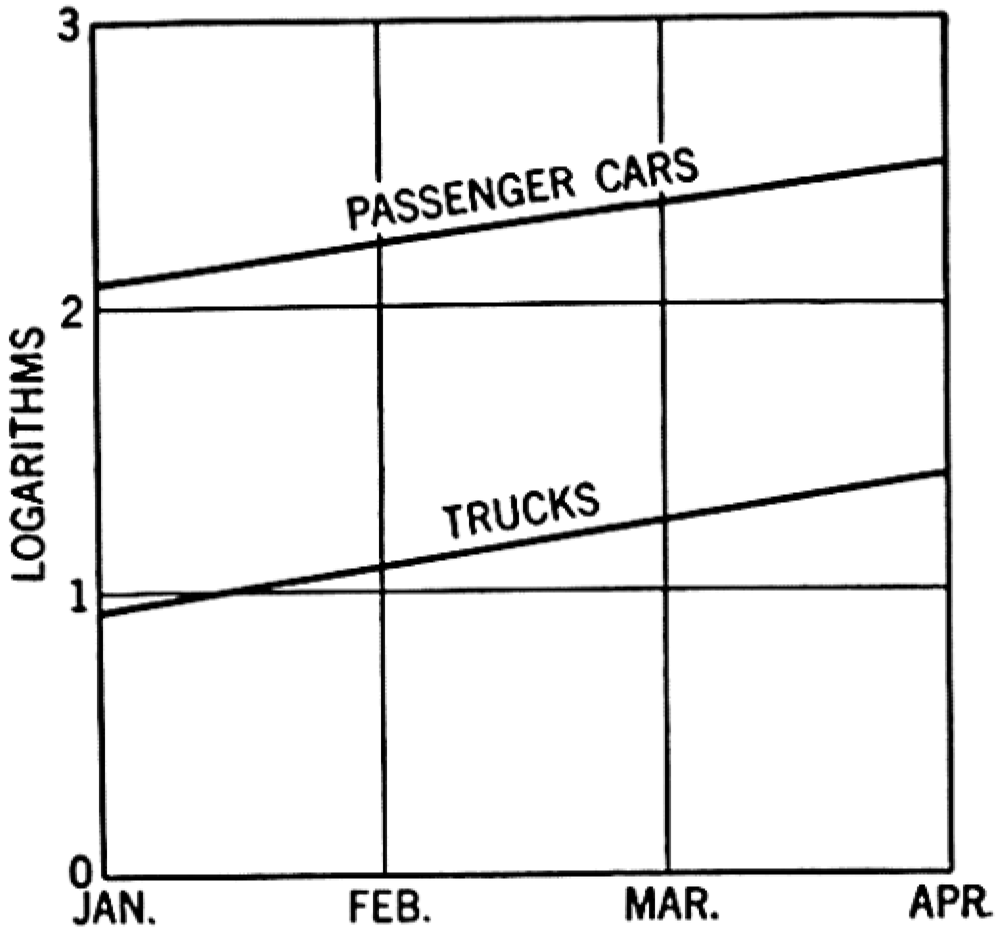

Figure 7. Rates of increase in the sales of passenger cars and trucks over a four-month period (ordinary graph paper).

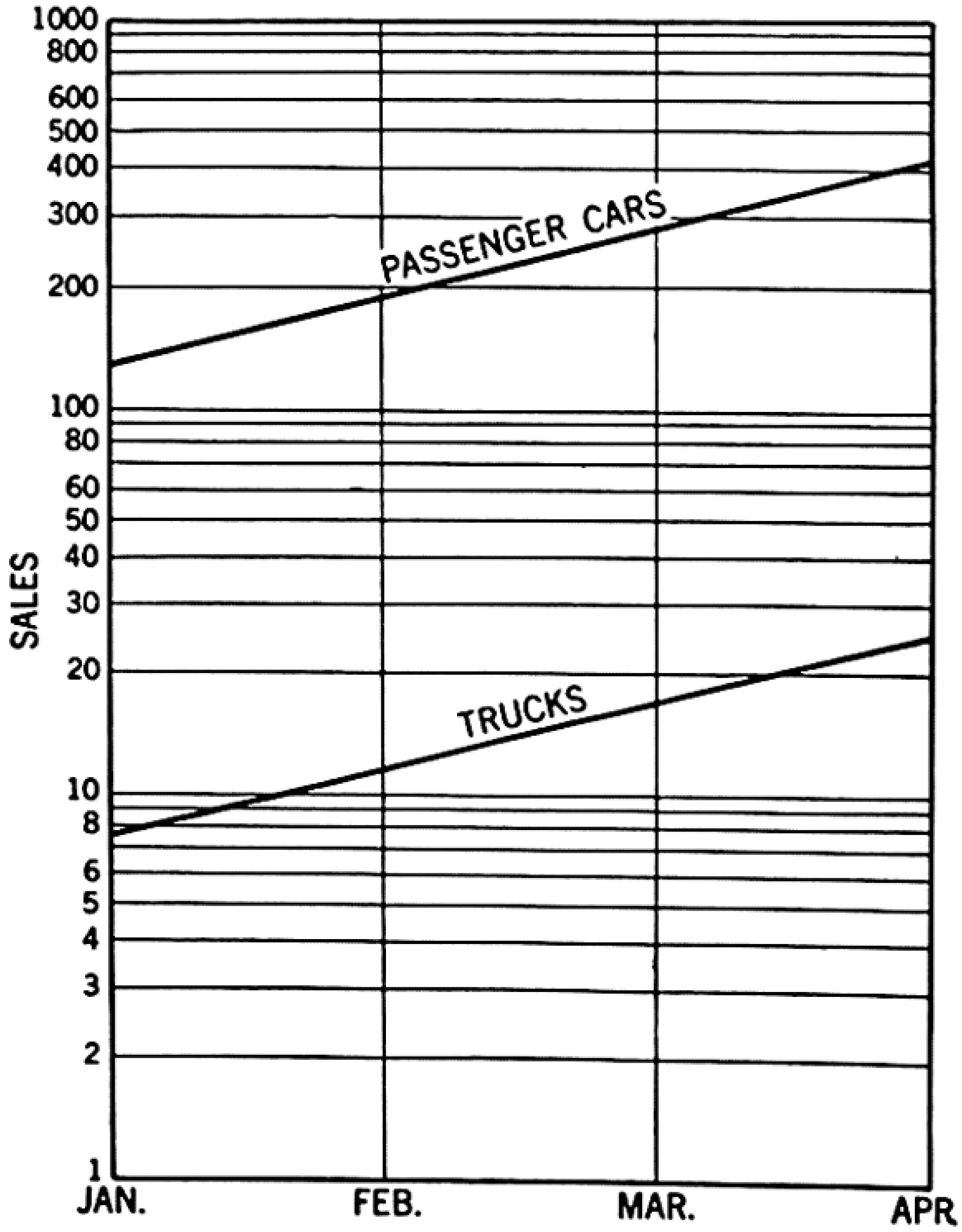

Figure 8. Rates of increase in the sales of passenger cars and trucks over a four-month period (semi-logarithmic graph paper).

PROBLEMS

1. Refer to Exercise 2 below and compare the growth of \$100 at $1\frac{1}{2}$ per cent *compound interest* and the growth of \$500 at the same rate of interest by graphing the accumulated amounts for a period of 12 years. (*a*) Use ordinary graph paper. (*b*) Use semilogarithmic graph paper.

2. The U.S. Department of Commerce reports in *Business Statistics* for 1953 the following monthly average amounts of new group and wholesale insurance written for the years 1947 to 1952. Use a logarithmic graph to depict the growth of new group and wholesale insurance written.

Year	*Group and Wholesale Insurance, Millions of Dollars*
1947	242
1948	260
1949	252
1950	517
1951	351
1952	420

3. The Clean-Rite Sweeper Company kept a record of the number of household demonstrations of their product and the number of resulting sales for the first four months of the year. Use a logarithmic graph, and compare the relative rates of increase. Interpret your results.

Month	*Number of Demonstrations*	*Number of Sweepers Sold*
Jan.	40	403
Feb.	56	487
Mar.	79	552
Apr.	112	686

4. Show the comparative rates of increase in the population of the United States from decade to decade by a logarithmic graph based on the following data:

Year	*Population*	*Year*	*Population*
1870	38,558,371	1920	105,710,620
1880	50,155,783	1930	122,775,046
1890	62,947,714	1940	131,669,275
1900	75,994,575	1950	150,697,361
1910	91,972,266		

Exercises

1. Compute by using logarithms the values of the following expressions:

(*a*) $0.84 \times 85.50 \times 405.33$,

(*b*) $356.1 \times (1.08)^2$,

(*c*) $(1.03)^5 \times 890.50$,

(*d*) $\dfrac{-5.057}{346.42}$,

(*e*) $\dfrac{\sqrt[3]{47.83}}{25.35}$,

(*f*) $\sqrt[5]{\dfrac{58.95}{9.50}}$,

(*g*) $\dfrac{500(1.03)^{10}}{(1.04)^3}$,

(*h*) $16(0.625)^{21}$

(*i*) $0.7211\sqrt[3]{-72.01}$,

(*j*) $\dfrac{(-14.012)(2.65)^5}{\sqrt[3]{-0.0556}}$,

(*k*) $\sqrt[4]{\dfrac{463.89 \times 24.63 \times (1.36)^3 \times 7.5}{225 \times 97.5}}$,

(*l*) $(0.2)^{0.2}$,

(*m*) $(-0.4222)^{-\frac{2}{3}}$,

(*n*) $\dfrac{\log 413}{\log 0.36}$,

(*o*) $\dfrac{\log 3 \log 7}{\log 5}$.

2. The amount of money accumulated at compound interest is given by the formula

$$S = P(1 + r)^n$$

in which S = compound amount, P = principal, r = rate of interest per conversion period, and n = number of conversion periods. Find the value of S if $P = \$2{,}500$, $r = 0.02$, and $n = 8$. (*Hint:* Add 0.02 to 1 before using logarithms, for logarithms cannot be used to subtract and add.)

3. If $3^x = 243$, find x.

Solution. Take logarithms of each side,

$$\log 3^x = \log 243$$

Then $$x \log 3 = \log 243 \quad \text{(from Theorem III)},$$

and $$x = \frac{2.38561}{0.47712} = 5.$$

Note: Logarithms are themselves numbers and may be divided, multiplied, and so forth, just like any other numbers.

4. Given the formula $L = ar^{n-1}$, in which $L = 26$, $a = 7$, and $n = 7$. Find r.

5. Given the formula $S = \dfrac{ar^n - a}{r - 1}$, in which $a = 20$, $r = 3.2$, and $n = 15$. Find S.

6. The present value A of an amount S due after n conversion periods where the interest rate is r per conversion period is given by the formula $A = \dfrac{S}{(1+r)^n}$. Find A if $S = \$1{,}200$, $r = 0.03$, and $n = 8$.

7. The amount of an annuity certain of \$1 per year for n years at accumulated compound interest is given by the formula $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$. Solve for n if $s_{\overline{n}|i} = 3.1525000$ and $i = 0.05$.

Solve for the unknown in each of the following problems:

8. $(1+i)^{27} = 1.947800.$

9. $(1.045)^n = 3.4297000.$

10. $(1+i)^{12} = 1.30605.$

11. $6.1^x = 73.4.$

12. $(1.06)^{-n} = 0.2198100.$

13. $(1+i)^{-36} = 0.6116300.$

14. $x^{7.2} = 72.$

15. $4^{3+x} = 23.62.$

16. $5(7)^{x^2} = 4716.$

17. $(2x)^{1.1} = 7.03.$

18. $85 = 5(r)^{15}.$

19. $y = \dfrac{(0.1)^{0.1}}{0.1} \log 0.1.$

20. $w = (-3)^{\frac{2}{3}} \log 3.$

Self-Test

Time: 50 Minutes

(Score 5 points for each problem correctly solved)

The student should solve Problems 1 to 15 without referring to any source of information other than the accompanying portion of the table of logarithms. For Problems 16 to 20, Table I should be used.

N	0	1	2	3	4	5	6	7	8	9
560	74819	827	834	842	850	858	865	873	881	889
561	74896	904	912	920	927	935	943	950	958	966
562	74974	981	989	997	*005	*012	*020	*028	*035	*043
563	75051	059	066	074	082	089	097	105	113	120

1. $\log 562 = ?$

2. $\log 0.0561 = ?$

3. $\log 0.05636 = ?$

4. $\log (56.32)^{\frac{1}{3}} = ?$

5. $\log 56.275 = ?$

6. $\log 0.00056053 = ?$

7. $\log (561)(563) = ?$

8. $\log (5{,}623 \div 5.6) = ?$

9. $\log (5.62)^4 - \log (0.563)^{\frac{1}{2}} = ?$

10. $\log 5628 \div \log 56.19 = ?$

11. If $\log N = 2.74908$, then $N = ?$

12. If $\log N = 8.75023 - 10$, then $N = ?$

13. If $\log N = \overline{2}.75092$, then $N = ?$

14. If $\log N = 7.74834$, then $N = ?$

15. If $\log N^2 = 1.49824$, then $N = ?$

16. Find the cube root of 1.025.

17. Find the value of $\frac{19.840 \times 120 \times 0.02}{365}$.

18. Find the value of x in the proportion

$$\frac{x}{297.50} = \frac{784}{1{,}728}.$$

19. Given $S = P(1 + r)^n$. Find S when $P = 8{,}500$, $r = 0.02$, $n = 8$.

20. Given $i = \left(1 + \frac{j}{m}\right)^m - 1$. Find the value of i when $j = 0.03$ and $m = 12$.

Chapter VIII

PROGRESSIONS AND PROBLEMS OF FINANCE

82. Arithmetic Series. A man agrees to pay a debt in 6 monthly installments. The first payment is to be \$10, and each month the payments are to increase by \$5. What is the total amount to be paid?

The 6 payments whose sum we wish to compute are seen to be \$10, \$15, \$20, \$25, \$30, and \$35, and the sum of these payments is

$$\$10 + \$15 + \$20 + \$25 + \$30 + \$35 = \$135.$$

Observe that any one of the payments after the first is found by adding \$5 to the one before it—that is, if we add \$5 to the first payment, we get the second payment, and if we add \$5 to the second payment, we get the third payment, and so forth. Such a set of quantities, or numbers, in which any term of the set, except the first, may be obtained from the immediately preceding one by a definite rule is called a *sequence*, or *progression*. The indicated sum of a sequence of terms is called a *series*. When the rule is to add the same constant to a given term to obtain the next term, the sequence is called an *arithmetic progression* and the sum of the terms is called an *arithmetic series*.

Arithmetic progressions are commonly used today in a variety of situations. Tables of weights and measures, tables of simple interest on a note, tables giving prices of sugar or coal, and many others depend upon adding equal amounts. In fact, many problems arising in business may be solved quite easily by the use of progressions. For this reason we shall find it convenient to derive formulas which will be useful for our purposes. We shall introduce the following symbols in terms of which we shall later write our formulas:

a = first term (\$10, in the example above),

d = common difference (\$5, in the example above),

n = number of terms (6, in the example above),

L = last term (\$35, in the example above),

S = sum of n terms (\$135, in the example above).

(1) **Formula for the Last Term.** In order to fix clearly in mind the way in which certain of these symbols enter into a general expression for the last term (or nth term) of an arithmetic series, we shall rewrite the sum of the series of the foregoing example and place directly beneath it the corresponding symbolic expression. We have

$$\begin{aligned} 135 &= 10 + (10 + 5) + (10 + 2 \times 5) + \cdots + (10 + 5 \times 5), \\ S &= a + (a + d) + (a + 2d) + \cdots + (a + 5d), \\ \text{Sum} &= (\text{First Term}) + (\text{Second Term}) + \cdots + (\text{Sixth Term}). \end{aligned}$$

It is seen that each term contains a. The second term contains $1d$, the third $2d$, the fourth $3d$, and so forth, and finally the sixth term contains $5d$. If there had been 9 terms instead of 6, we could expect the last term to contain $8d$. In general, the last term would contain as many d's as one less than the number of terms—that is, $(n - 1)d$. Hence, the formula for the last term (nth term) of an arithmetic progression is

$$L = a + (n - 1)d. \tag{1}$$

In our example we may verify our earlier calculations by noting that $a = 10$, $n = 6$, $d = 5$ and observing that substitutions into the formula yield

$$35 = 10 + (6 - 1)5.$$

Illustration 1: Find the 12th term of the sequence 4, 7, 10, 13, ...

We see that $a = 4$, $d = 3$, $n = 12$. When we substitute these values into formula (1), we find

$$L = 4 + (12 - 1)3 = 4 + 11 \times 3 = 4 + 33 = 37.$$

PROBLEM

Find the 8th term of the sequence 3, 7, 11, 15, ...
Answer: 31.

Illustration 2: Form an arithmetic progression, given the elements $a = 100$, $d = -6$, $n = 4$.

The 4 terms to be found are a, $a + d$, $a + 2d$, and $a + 3d$. Hence, the desired terms are 100, 94, 88, 82.

PROBLEM

Form an arithmetic progression, given $a = 12$, $d = 5$, $n = 11$.
Answer: 12, 17, 22, . . . , 62.

Illustration 3: The first term of an arithmetic series of 7 terms is 50, and the last term is 62. What is the *common difference* between successive terms?

Here, we have $L = 62$, $a = 50$, $n = 7$, and we desire to find d. Upon substituting into formula (1), we have

$$62 = 50 + 6d.$$

Solving for d (transposing, collecting, and dividing by 6), we find $d = 2$.

PROBLEM

In an arithmetic series of 6 terms, the first term is 5, and the last term is 35. Find the common difference between each term.

Answer: 6.

(2) **Formula for the Sum of n Terms.** Let us now return to the example with which we opened this section. We have

$$S = 10 + 15 + 20 + 25 + 30 + 35.$$

Reversing the order of the terms, we may write the equivalent sum

$$S = 35 + 30 + 25 + 20 + 15 + 10.$$

Writing these two expressions in forms involving the common difference, we have

$$S = 10 + (10 + 5) + (10 + 2 \times 5) + (10 + 3 \times 5) + (10 + 4 \times 5) + (10 + 5 \times 5),$$

and

$$S = 35 + (35 - 5) + (35 - 2 \times 5) + (35 - 3 \times 5) + (35 - 4 \times 5) + (35 - 5 \times 5).$$

Adding corresponding members of the two equations, we notice that in each term the part involving the common difference drops out because of the opposite signs, and we are left with

$$2S = (10 + 35) + (10 + 35) + (10 + 35) + (10 + 35) + (10 + 35) + (10 + 35),$$

or
$$2S = 6(10 + 35),$$

from which
$$S = \frac{6(10 + 35)}{2} = 135.$$

Since $n = 6$, $a = 10$, and $L = 35$, it appears that, in general, the formula for S is $S = \frac{n(a + L)}{2}$. But to prove that this is the case, we employ the general expression of n terms involving the symbols. We have for our two equivalent sums, the first of which is written in a forward order and the second of which is written in the reverse order,

$$S = a + (a + d) + (a + 2d) + (a + 3d) + \cdots + (L - d) + L.$$
$$S = L + (L - d) + (L - 2d) + (L - 3d) + \cdots + (a + d) + a.$$

Adding the two expressions for S, we notice that the terms in d drop out, and we have

$$2S = (a + L) + (a + L) + (a + L) + \cdots + (a + L),$$

or $2S = n(a + L)$ since we get an $(a + L)$ for each of the n terms of the combined series. Finally, solving for S, we find

$$S = \frac{n(a + L)}{2}. \tag{2}$$

Since we already have a formula for L, we may obtain an alternative formula for S by substituting the value of L from formula (1) into formula (2). We have $S = \frac{1}{2}n(a + L)$ and $L = a + (n - 1)d$. Making the substitution, we have

$$S = \frac{n}{2}[a + a + (n - 1)d],$$

or
$$S = \frac{n}{2}[2a + (n - 1)d]. \tag{3}$$

To illustrate the use of formula (3) for our example dealing with the

6 monthly installments, we substitute the proper values of *a*, *n*, and *d* and find

$$\begin{aligned} S &= \tfrac{6}{2}[2(10) + (6 - 1)5] \\ &= 3(20 + 25) = 3 \times 45 = 135. \end{aligned}$$

Illustration 1: In constructing a building 30 stories high it was found that the cost of certain equipment increased $150 with each story. If the cost of the equipment was $1,800 for the first story, what was the entire cost of the equipment?

Here, $a = 1{,}800$, $d = 150$, and $n = 30$. Substituting these values into formula (3), we find

$$\begin{aligned} S &= \tfrac{30}{2}[2 \times 1{,}800 + (30 - 1)150] \\ &= 15(3{,}600 + 4{,}350) = 15 \times 7{,}950 = \$119{,}250. \end{aligned}$$

PROBLEM

A boy started a Christmas fund, saving 5¢ the first week, 10¢ the second, 15¢ the third week, and so forth, for 50 weeks. How much did he save?

Answer: $63.75.

Illustration 2: A manufacturing company installs a machine at a cost of $1,400. At the end of 9 years the machine has a value of $200. Assuming that the yearly depreciation is a constant amount, find the annual depreciation.

Here, $a = 1{,}400$, $n = 10$, and $L = 200$. Our problem is to find *d*. Formula (1) contains all our given elements and the quantity we seek; hence, we shall substitute into that formula and solve for *d*. We have

$$\begin{aligned} L &= a + (n - 1)d, \\ 200 &= 1{,}400 + (10 - 1)d, \\ 200 &= 1{,}400 + 9d. \end{aligned}$$

Transposing, collecting, and dividing by the coefficient of *d*, we find $d = -\$133.33$. That is to say, the annual depreciation is $133.33.

PROBLEM

A housewife paid \$67.50 for an electric sweeper. At the end of 10 years it was valued at \$10. Assuming the machine depreciated the same amount each year, what was the annual depreciation?

Answer: \$5.75.

Illustration 3: A debt of \$200 is to be paid in yearly installments of \$40 plus the accrued interest on the debt at 5%. How much interest must be paid in discharging the debt?

Since the debt is reduced by equal amounts each year, the interest payments would be as follows:

$$\$200 \times 0.05 = \$10 \text{ interest 1st year,}$$

$$\$160 \times 0.05 = \$\ 8 \text{ interest 2nd year,}$$

$$\$120 \times 0.05 = \$\ 6 \text{ interest 3rd year,}$$

$$\$\ 80 \times 0.05 = \$\ 4 \text{ interest 4th year,}$$

$$\$\ 40 \times 0.05 = \$\ 2 \text{ interest 5th year.}$$

We see that it will take 5 years ($\frac{200}{40}$) to pay off the debt in full. Furthermore, the interest payments are \$2 less each year and, therefore, constitute an arithmetic progression. We have $a = 10$, $d = -2$, $n = 5$. Employing formula (3) for the sum of the terms of an arithmetic series, we find

$$S = \tfrac{5}{2}[2 \times 10 + (5 - 1)(-2)] = \tfrac{5}{2}[20 - 8] = \tfrac{5}{2} \times 12 = \$30.$$

PROBLEM

A college debt of \$600 was paid in yearly installments of \$100, and the interest rate was 3%. How much interest was paid?

Answer: \$63.

Exercises

Find L and S for the following progressions:

1. 2, 9, 16, . . . ($n = 12$).
2. 23, 21, 19, 17, . . . ($n = 12$).
3. 21, 15, 9, . . . ($n = 10$).
4. 8, 3, −2, . . . ($n = 11$).
5. −2, −15, −28, . . . ($n = 10$).
6. 3, 11, 19, . . . ($n = 20$).

7. 0.3, 0.7, 1.1, . . . $(n = 50)$.

8. $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \ldots$ $(n = 16)$.

9. $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots$ $(n = 15)$.

10. Given $a = 2$, $d = 1\frac{1}{2}$, $L = 20$, find n and S.

11. Given $a = \frac{1}{3}$, $d = \frac{1}{2}$, $L = \frac{29}{6}$, find n and S.

12. The 25th term of an arithmetic progression is 100; the common difference is $\frac{1}{7}$; find a and S.

13. Given $a = 25$, $S = 105$, $n = 5$, find d and the 5th term.

14. Given the 6th term as 4, the sum of 6 terms as 42, find a and d.

15. Given $a = 2$, $L = 20$, $S = 55$, find n and d.

16. The 3rd term of an arithmetic progression is 13, and the 10th term is 55; find the 6th term.

17. Find a value of x for which 1, $2 + 3x$, $4 - 2x$ will be in arithmetic progression.

18. Given $a = -1$, $L = 2$, $S = 55$, find n and d.

19. Given $a = 5$, $n = 8$, $S = 124$, find L and d.

20. Express d in terms of L, a, and n. [*Hint:* Solve formula (1) for d.]

21. Form an arithmetic progression, given the following elements: (*a*) $d = 7$, $n = 6$, $a = 13$; (*b*) $a = 3$, $d = -2$, $n = 5$; (*c*) $S = 64$, $L = 24$, $n = 8$.

22. An employee started work for a company at \$3,200 for the first year and received a regular yearly increase of \$200 for the next 5 years. What was his salary at the end of 6 years' service with the company? How much did the company pay him in the course of 6 years?

23. A city issued bonds in the amount of \$50,000, bearing interest at $4\frac{1}{2}\%$, and the principal was to be repaid in 20 equal annual installments. Compute the amount of interest paid by the city during the life of the bonds.

24. A man invests his savings in the shares of a finance organization, depositing \$200 the first year. At the beginning of the second year he is credited with \$12 interest and deposits \$188. At the beginning of the third year his interest credit amounts to \$24, and he deposits \$176. If he continues to receive credit and to deposit savings in this manner, how much interest has he earned at the end of 5 years? How much cash has he paid in?

25. A carpenter wishes to purchase a length of rod sufficient for 12 rungs of a ladder. If the rungs diminish uniformly from 20 in. at the base to $11\frac{3}{4}$ in. at the top, what length rod must he buy?

26. A father starts a savings fund for his son by putting in 10¢ the first week, 15¢ the second week, 20¢ the third week, and so on, for 52 weeks. How much is saved after 52 weeks?

27. A student borrowed \$800 to meet his expenses for 1 year at college. After 3 months he paid \$10 in full payment of the interest that had accumulated. What rate of simple interest was he paying?

28. In a game of pool each ball counts the number it is marked, and the 15 balls are numbered consecutively from 1 to 15. One player "ran" all the balls from 1 to 7, inclusive, and his opponent "ran" the remaining balls. How many points did each receive?

29. A man borrowed a sum of money at the rate of 2% per year. When the note matured, he paid $267.50. How much money did he borrow if the note was paid in 3 years and 6 months?

30. The Auto Finance Company loans $200 to be paid back in 10 monthly installments of $21.33. Would it be cheaper for the borrower to repay it in 10 monthly installments of $20 plus simple interest at 12% on the unpaid balance? How great is the difference?

31. A punch board has 200 numbers on it (Nos. 1 to 200). If the customer pays the amount of the number punched in cents, how much revenue is derived from the board?

32. A clock strikes the hour (that is, at 3 o'clock it strikes 3 times, and so forth). How many times does it strike in a day and a night? How many times does it strike if it strikes once for the half hour also?

33. In a potato race, 10 potatoes are placed one by one along a straight path at equal intervals, the first interval being the distance between the starting tape and the first potato. What should be the length of the intervals if a contestant starting at the tape travels $\frac{1}{2}$ mile in returning the potatoes, one by one, to the tape?

34. A debt of $1,000 is paid in 8 annual installments of $125 each, plus interest on the unpaid balance at 6%. The first payment is made 1 year after the debt is incurred. What is the total amount paid in discharging the debt?

83. Geometric Series. A stockholder has $100 invested in a certain enterprise. Each year he makes 10% on his capital and buys additional stock with the earnings. How much stock does he own at the end of 4 years? What is the amount of the accumulated interest?

The amount of stock owned at the end of each year and the interest for each year may be calculated as follows:

$$\begin{array}{rl}
\$100 & \\
10 & \\
\hline
110 & \text{Stock, end 1st year} \\
11 & \\
\hline
121 & \text{Stock, end 2nd year} \\
12.10 & \\
\hline
133.10 & \text{Stock, end 3rd year} \\
13.31 & \\
\hline
146.41 & \text{Stock, end 4th year}
\end{array}$$

$\$100 \times 0.10 = \$10$ (interest 1st year),
$\$110 \times 0.10 = \$11$ (interest 2nd year),
$\$121 \times 0.10 = \$12.10$ (interest 3rd year),
$\$133.10 \times 0.10 = \$13.31$ (interest 4th year).

In this example the amount of stock owned at the end of each year may also be found by multiplying the amount owned at the end of the preceding year by 1.10. We have

$$\$100 \times 1.10 = \$110,$$

$$\$110 \times 1.10 = \$121,$$

$$\$121 \times 1.10 = \$133.10,$$

$$\$133.10 \times 1.10 = \$146.41.$$

Indicating the successive multiplication by which the stock each year is found, we may write these amounts in the following manner:

$$\$100,\ \$100(1.10),\ \$100(1.10)^2,\ \$100(1.10)^3,\ \$100(1.10)^4.$$

Such a sequence, in which each term except the first is found by multiplying the preceding term by the same constant, is called a *geometric progression*, and the indicated sum of these terms is called a *geometric series.*

(1) **Formula for the Last Term.** The preceding progression expressed in symbols could be written thus,

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,$$

where a stands for the first term and r for the common multiplier, or *ratio.* It will be noticed that the exponent of the ratio for the 5th term is 4. If there had been 7 terms, the exponent for the 7th term would have been 6; for 13 terms, the exponent 12, and so forth. In general, a geometric progression of n terms would be written

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,\ \ldots,\ ar^{n-1} \qquad (n \text{ terms}).$$

Hence, the formula for the last term (L) of a geometric progression of n terms is

$$L = ar^{n-1}. \tag{1}$$

In our example we may use this formula to obtain the amount of stock owned at the end of 4 years. We have $a = 100$, $r = 1.10$, and $n = 5$. Substituting into formula (1), we find

$$L = 100 \times (1.10)^4 = 100 \times 1.4641 = \$146.41.$$

The work involved in using formula (1) could have been performed by using logarithms. For our example,

$$L = 100 \times (1.10)^4.$$

Taking logarithms, we obtain

$$\log L = \log [100 \times (1.10)^4],$$

$$\log L = \log 100 + \log (1.10)^4 \qquad \text{(from rule for logarithm of a product),}$$

$$\log L = \log 100 + 4 \log (1.10) \qquad \text{(from rule for logarithm of a power),}$$

$$\log L = 2.00000 + 4 \times 0.04139 \qquad \text{(from table of logarithms),}$$

$$\log L = 2.00000 + 0.16556 = 2.16556.$$

Hence, after referring to a table of logarithms, we find $L = \$146.41$.

Illustration 1: Find the 9th term of the geometric progression 3, 6, 12, 24, ...

Here, $a = 3$, $r = 2$, and $n = 9$. We have

$$L = ar^{n-1} = 3 \times 2^8 = 3 \times 256 = 768.$$

PROBLEMS	*Answers:*
Find the last term of the sequence:	
1. $1, \frac{1}{4}, \frac{1}{16}, \ldots$ to 8 terms.	1. 1/16,384.
2. $\frac{2}{3}, 2, 6, \ldots$ to 6 terms.	2. 162.
3. $3, -9, 27, \ldots$ to 7 terms.	3. 2,187.
4. $1.06, 1.1236, 1.191016, \ldots$ to 5 terms.	4. $(1.06)^5$.

The formula for the last term of a geometric progression involves four quantities. When any three of them are known, we may always solve for the fourth.

Illustration 2: Solve $L = ar^{n-1}$ for a in terms of L, r, and n.

Transposing terms and dividing through by the coefficient of a, we have

$$a = \frac{L}{r^{n-1}}$$

PROBLEM

Solve the formula for the last term of a geometric progression for r; for n.

Answer:

$$r = (L/A)^{\frac{1}{n-1}}$$

$$n = 1 + \frac{\log L - \log a}{\log r}$$

(2) **Formula for the Sum of n Terms.** The second question in the example at the beginning of this section calls for the amount of the accumulated interest. We have seen that these interest payments for the 4 years were \$10, \$11, \$12.10, and \$13.31, respectively. Thus, letting S stand for the sum, we have

$$S = 10 + 11 + 12.10 + 13.31.$$

Of course, by simply adding the four amounts we find the total amount of accumulated interest to be \$46.41. This agrees, as it should, with the difference between the stock owned at the end of the fourth year and the amount of the initial investment. But for a great number of problems that arise in financial situations, as, for example, in compound interest, we shall wish to employ formulas that will enable us to compute rapidly how large a sum is accumulated at the end of a given number of years or interest periods. Can we arrive at a formula for this purpose?

Since each term in the expression for S directly above is equal to 1.10 times the preceding term, we may write

$$S = 10 + 10(1.10) + 10(1.10)^2 + 10(1.10)^3. \qquad (A)$$

Now, suppose we multiply both sides of this equation by 1.10. We then have

$$1.10S = 10(1.10) + 10(1.10)^2 + 10(1.10)^3 + 10(1.10)^4. \qquad (B)$$

If we subtract (A) from (B), certain terms of the right-hand members cancel and we have

$$1.10S - S = 10(1.10)^4 - 10.$$

We may take S out as a factor on the left and write

$$S(1.10 - 1) = 10(1.10)^4 - 10,$$

or

$$S = \frac{10(1.10)^4 - 10.}{1.10 - 1}$$

For the part of our example dealing with interest payments, we have, using symbols, $a = 10$, $n = 4$, and $r = 1.10$. Hence, it would seem that our formula is

$$S = \frac{ar^n - a}{r - 1}$$

But to prove that we have here a correct formula, we shall use the general expression of n terms involving the symbols and carry out the work as above. We write

$$S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}, \qquad (A')$$

and

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n, \qquad (B')$$

formula (B') being formed by multiplying each member of formula (A') by r. Upon subtracting formula (A') from (B'), there results

$$rS - S = ar^n - a.$$

Factoring,

$$S(r - 1) = ar^n - a.$$

Dividing through by $(r - 1)$, $$S = \frac{ar^n - a}{r - 1}. \qquad (2)$$

Formula (2), therefore. gives the sum of n terms of a geometric progression.

Illustration 1: Find the last term and the sum of the sequence 4, 12, 36, . . . to 6 terms.

Here, $a = 4$, $r = 3$, and $n = 6$.

$$L = ar^{n-1} = 4 \times 3^{6-1} = 4 \times 3^5 = 4 \times 243 = 972$$ (last term in series).

$$S = \frac{ar^n - a}{r - 1} = \frac{4 \times 3^6 - 4}{3 - 1} = \frac{4 \times 729 - 4}{2} = \frac{2{,}916 - 4}{2} = 1{,}456$$ (sum of 6 terms).

PROBLEMS

Find the last term and the sum of the sequence:

1. 7, 14, 28, . . . to 6 terms.
2. 100, 150, 225, . . . to 5 terms.

Answers:

1. 224; 441.
2. $506\frac{1}{4}$; $1{,}318\frac{3}{4}$.

Illustration 2: Sheet music sales of the Taylor Music Store for 1952 totaled $5,000. Because of better business conditions their records showed total sales of $6,298.56 at the close of the year 1955. Assuming that the sales increased each year over the preceding year at a constant rate, find the yearly per cent of increase and the total volume of sales for the 4 years.

Our first problem is to find r in a geometric sequence of four terms beginning with 5,000 and ending with 6,298.56. We write

$$a = 5{,}000 \qquad L = 6{,}298.56,$$
$$n = 4, \qquad r = ?$$

$L = ar^{n-1}$; hence, $6{,}298.56 = 5{,}000 \times r^3$, or $r^3 = \frac{6{,}298.56}{5{,}000}$. By logarithms,

$\log r^3 = \log 6298.56 - \log 5{,}000$ (from rule for logarithm of a quotient),

$3 \log r = \log 6298.56 - \log 5{,}000$ (from rule for logarithm of a power),

$3 \log r = 3.79924 - 3.69897 = 0.10027, \qquad \log r = 0.03342,$
and $r = 1.08.$

The sales for each year, therefore, are found to be 1.08 times those for the preceding year. Hence, there is a yearly increase of 8%.

For the second problem in the illustration, we use the formula for S in order to find the total sales for the 4 years. We have

$$\begin{aligned} S &= \frac{ar^n - a}{r - 1} \\ &= \frac{5{,}000 \times (1.08)^4 - 5{,}000}{1.08 - 1} = \frac{5{,}000 \times 1.360489 - 5{,}000}{0.08} \\ &= \frac{1{,}802.445}{0.08} = \$22{,}530.56, \text{ total sales.} \end{aligned}$$

PROBLEM

If the population of a city increases from 225,000 to 350,000 in 5 years, what is the average rate of yearly increase?

Answer: 9.2% (approximately)

Illustration 3: Find the sum of 8 terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Here, $a = 1$, $r = \frac{1}{2}$, and $n = 8$. We have

$$S = \frac{ar^n - a}{r - 1} = \frac{1 \times (\frac{1}{2})^8 - 1}{\frac{1}{2} - 1} = \frac{\frac{1}{256} - 1}{-\frac{1}{2}} = \frac{-\frac{255}{256}}{-\frac{1}{2}} = \frac{255}{256} \times \frac{2}{1}$$
$$= \frac{255}{128} = 1\,\frac{127}{128}.$$

PROBLEMS	*Answers:*
Find the sum of 6 terms of the series:	
1. $1 + \frac{1}{5} + \frac{1}{25} + \cdots$.	1. $\frac{3{,}906}{3{,}125}$.
2. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.	2. $\frac{182}{243}$.

(3) Formula for the Sum, when the Number of Terms Is Unlimited. In Illustration 3, we observe that the terms become smaller and smaller as the series progresses. This is always the case when the ratio is numerically less than 1. In fact, if the series were unlimited we could find a term numerically smaller than any specified positive value by going far enough along in the series. For example, when $n = 13$ in this series, the term would be $(\frac{1}{2})^{12}$, or approximately 0.00024. For an n of 25, we would have $(\frac{1}{2})^{24}$, or approximately 0.000000059604. For larger values of n, we would find still smaller terms. This behavior can be expressed by saying that the terms of this geometric series approach 0 as a *limit* as the number of terms becomes infinite. We may say, therefore, that when r is numerically less than 1, the quantity r^n approaches 0 as n becomes infinite. Hence, as we let n increase without limit, the formula

$$S = \frac{ar^n - a}{r - 1},$$

becomes

$$S = \frac{a(0) - a}{r - 1},$$

or

$$S = \frac{-a}{r - 1},$$

which, after changing the signs in the numerator and denominator, may be written

$$S = \frac{a}{1 - r}. \tag{3}$$

Formula (3) expresses the *limit of the sum* of an infinite geometric series. This limit is usually referred to simply as the *sum* of the infinite series.

Illustration 1: Find the limit of the sum, as n is increased infinitely, of the series:

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots.$$

We have $r = \frac{1}{2}$, $a = 1$. By formula (3)

$$S = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

The student should add together a sufficient number of the terms of this series to convince himself that the sum can be made as near to 2 as may be desired by taking enough terms.

PROBLEM

Find the sum of the infinite series $1 + \frac{1}{3} + \frac{1}{9} + \cdots$.
Answer: $\frac{3}{2}$.

Illustration 2: Find the sum of the infinite series $(1.025)^{-1} + (1.025)^{-2} + (1.025)^{-3} + \cdots$.

Here, $$a = r = (1.025)^{-1} = \frac{1}{1.025} = \frac{40}{41}.$$

Hence, $$S = \frac{a}{1 - r} = \frac{\frac{40}{41}}{1 - \frac{40}{41}} = \frac{\frac{40}{41}}{\frac{1}{41}} = 40.$$

PROBLEM

Find the sum of the infinite series $(1.02)^{-1} + (1.02)^{-2} + (1.02)^{-3} + \cdots$.

Answer: 50

Exercises

Find L and S in the following progressions:

1. 1, 3, 9, 27, . . . (5 terms).

2. $1, \frac{1}{4}, \frac{1}{16}, \ldots$ (7 terms).

3. 5, 10, 20, . . . (8 terms).

4. $1, -2, 4, \ldots$ (6 terms).

5. 5, 15, 45, . . . ($n = 5$).

6. 3, 6, 12, 24, . . . ($n = 9$).

7. $3, -12, 48, \ldots$ ($n = 7$).

8. 16, 8, 4, 2, . . . ($n = 7$).

9. $\frac{2}{3}, \frac{2}{7}, \frac{6}{49}, \ldots$ ($n = 5$).

10. 5, 0.5, 0.05, 0.005, . . . ($n = 7$).

11. 172, 344, 688, . . . ($n = 10$).

12. $1.04, (1.04)^2, (1.04)^3, \ldots$ ($n = 5$).

13. $-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \ldots$ ($n = 8$).

14. $-6, -30, -150, \ldots$ ($n = 7$).

15. $5, 5\sqrt{3}, 15 \ldots$ ($n = 12$).

16. Given $a = 2$, $r = 1\frac{1}{2}$, $L = 10\frac{1}{8}$, find n and S.

17. Given $a = 5$, $L = 405$, $n = 5$, find r and S.

18. The 5th term of a geometric progression is 8; the common ratio is $\frac{2}{3}$; find a and S.

19. Given $a = 42$, $r = 0.1$, $S = 46.662$, find n and L.

20. The 3rd term of a geometric progression is 4; the 7th term is 36; find the 8th term.

21. Find x so that $x - 1$, $x + 2$, $x + 9$ will form a geometric progression.

22. Find S and L, when $a = 5$, $r = \frac{3}{4}$, $n = 25$.

23. Find r and S, when $a = 3$, $n = 9$, $L = 640$.

24. Find n and S, when $a = 130$, $L = 10{,}530$, $r = 3$.

25. Find S and L, when $a = 24$, $n = 10$, $r = \frac{1}{4}$.

26. Solve $L = ar^{n-1}$ for (*a*) a in terms of r, n, L, (*b*) r in terms of n, a, L, (*c*) n in terms of a, r, L.

27. Solve $S = \dfrac{ar^n - a}{r - 1}$ for (*a*) a in terms of r, S, n, (*b*) n in terms of r, a, S.

28. On a certain radio quiz program each contestant was given a prize according to the number of correct responses made. On one particular program the sponsors paid out \$2, \$4, \$8, . . . and so forth, until six persons had received prizes. What was the total amount of money paid out?

29. In performing a certain division the quotient 0.272727 . . . was found, the numbers 2, 7 being repeated indefinitely. What fraction does this number represent? (*Hint:* Write the series $0.27 + 0.0027 + 0.000027 + \cdots$.)

30. Find the limit of the sum of an infinite number of terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$.

31. Find the sum of the infinite series $0.3 + 0.03 + 0.003 + \cdots$.

32. A certain manufacturing firm estimated that production would increase 20% each year over that of the preceding year. If the volume of business was $8,000 at the end of the year 1954, what amount of production could be expected in 1960?

33. Under certain conditions the number of bacteria in milk doubles every 3 hr. By how much will the number be multiplied at the end of 24 hr?

34. A machine installed in a certain steel plant cost $1,400. The depreciation for a month on the machine at the end of any month was computed at the rate of 5% of its value at the beginning of the month. How much value was placed on the machine at the end of 20 months? How much had the machine depreciated during the 20th month?

35. Los Angeles had approximately 6,000 people in 1870 and in 1940 had approximately 1,500,000. Find the average percentage increase per decade. If this rate of increase had continued until 1950, what would have been the population then? Compare this with the actual 1950 population of 1,970,000.

36. A building cost $500,000. At the end of each year, the owners deducted 10% of its carrying value as estimated at the beginning of the year. What is the estimated value at the end of 20 years?

37. The population of Washington, D.C., increased from 487,000 in 1930 to 663,000 in 1940. Find the average rate of yearly increase. Compare this rate with the average rate of yearly increase between 1940 and 1950 when the population was 802,000.

38. The stock of the Southeast Northwestern Railway fell steadily from 80 in 1925 to 5 on the same day of 1935. What was the average annual percentage decrease?

39. A rubber ball is dropped from a height of 6 ft and rebounds $\frac{1}{3}$ the distance from which it falls each time. How far has the ball traveled at the instant it hits the floor for the 6th time?

40. How far would the ball in Problem 39 travel if it continued to bounce indefinitely?

84. Compound Amount. A man deposits $100 in a savings account and draws 3% interest each year. He leaves the interest in his account and thus draws interest on the earnings, too, after the first year. Each year he has 1.03 times as much in the bank at the end of the year as he had at its beginning. How much has he in the account at the end of 5 years?

We may solve this problem and display calculations in detail as follows:

\$100	
3	Interest
103	Amount, end 1st year
3.09	Interest
106.09	Amount, end 2nd year
3.18	Interest
109.27	Amount, end 3rd year
3.28	Interest
112.55	Amount, end 4th year
3.38	Interest
115.93	Amount, end 5th year

$\$100 \times 0.03 = \$3.00$ (interest 1st year),

$\$103 \times 0.03 = \$3.09$ (interest 2nd year),

$\$106.09 \times 0.03 = \$3.18$ (interest 3rd year),

$\$109.27 \times 0.03 = \$3.28$ (interest 4th year),

$\$112.55 \times 0.03 = \$3.38$ (interest 5th year).

Observe that the amount at the end of each year equals the principal plus the interest for that year and that this amount is the principal for the next year. This relationship may be shown as follows:

Principal + Interest = Amount (written in factor form).

End 1st year: $100 + 100(0.03) = 100(1 + 0.03)$.

End 2nd year: $100(1 + 0.03) + 100(1 + 0.03)(0.03)$
$= 100(1 + 0.03)(1 + 0.03) = 100(1 + 0.03)^2$.

End 3rd year: $100(1 + 0.03)^2 + 100(1 + 0.03)^2(0.03)$
$= 100(1 + 0.03)^2(1 + 0.03) = 100(1 + 0.03)^3$.

End 4th year: $100(1 + 0.03)^3 + 100(1 + 0.03)^3(0.03)$
$= 100(1 + 0.03)^3(1 + 0.03) = 100(1 + 0.03)^4$.

End 5th year: $100(1 + 0.03)^4 + 100(1 + 0.03)^4(0.03)$
$= 100(1 + 0.03)^4(1 + 0.03) = 100(1 + 0.03)^5$.

From this display it appears that we could have computed the amount saved at the end of 5 years (compound amount) by evaluating the expression $100(1 + 0.03)^5$. Calling this amount S, we have

$$S = 100(1 + 0.03)^5 = 100 \times 1.159274$$

$$= 100 \times 1.1593 \quad \text{(rounding off)}$$

$$= \$115.93 \quad \text{(compound amount of \$100 at 3\% for 5 years).}$$

Using symbols so that we may establish the formula for S in terms of the principal P, the rate r per period of time, and the number of periods of time n, we show the calculations for the successive periods as follows:

$$\text{Principal } (P) + \text{Interest } (I) = \text{Amount } (S).$$

End 1st period: $P + Pr = P(1 + r)$,

End 2nd period:

$$P(1 + r) + P(1 + r)r = P(1 + r)(1 + r) = P(1 + r)^2,$$

End 3rd period:

$$P(1 + r)^2 + P(1 + r)^2 r = P(1 + r)^2(1 + r) = P(1 + r)^3,$$

and so forth, for n periods of time. In the Amount column it is seen that each term is $1 + r$ times the one just above it. The values of S constitute, therefore, a geometric progression where the *first term* is $P(1 + r)$, the *constant ratio* is $1 + r$, and the *number of terms* is n. By application of the formula $L = ar^{n-1}$, our amount at the end of n periods becomes

$$S = P(1 + r)(1 + r)^{n-1},$$

or

$$S = P(1 + r)^n. \tag{4}$$

Formula (4) is known as the *compound interest formula.* It involves the four elements S, P, r, and n. When, therefore, any three of these are known, the other can be found by substitution in the formula and solving for the unknown value. This formula is important in many problems in finance and insurance and wherever the accumulation of sums follows the law of geometric progression. *Unless otherwise stated, interest will be assumed to be compounded annually* in the problems that are to follow.

Illustrations:

1. A person deposited \$1,000 at interest at 2% compounded annually and left it to accumulate for 10 years. What amount was then due him?

We have

$P = 1{,}000,$ $\qquad S = P(1 + r)^n$ (formula for S),

$r = 0.02,$ $\qquad \log S = \log P + n \log (1 + r),$

$n = 10,$ $\qquad \log P = 3.00000,$

$S = ?$ $\qquad n \log (1 + r) = 0.08600,$

$$\log S = 3.08600,$$

$$S = \$1{,}219.$$

2. A person deposited a sum of money at interest for 10 years at 2% compounded annually and received \$1,219 at the end of the 10-year period. What sum did he deposit?

$S = 1{,}219,$ $\qquad S = P(1 + r)^n,$

$r = 0.02,$ $\qquad P = \dfrac{S}{(1 + r)^n}$ (formula solved for P),

$n = 10,$ $\qquad \log P = \log S - n \log (1 + r),$

$P = ?$ $\qquad \log S = 3.08600,$

$$n \log (1 + r) = 0.08600,$$

$$\log P = 3.00000,$$

$$P = \$1{,}000.$$

3. A person deposited \$1,000 at 2% interest compounded annually and got back \$1,219. How long was the money at interest?

$P = 1{,}000,$ $\qquad S = P(1 + r)^n,$

$S = 1{,}219,$ $\qquad \log S = \log P + n \log (1 + r),$

$r = 0.02,$ $\qquad n \log (1 + r) = \log S - \log P,$

$n = ?$ $\qquad n = \dfrac{\log S - \log P}{\log (1 + r)}$ (formula solved for n),

$$\begin{aligned} \log S &= 3.08600 \\ \log P &= \underline{3.00000} \\ &\ 0.08600 \end{aligned}$$ (difference)

$$\log (1 + r) = 0.00860,$$

$$n = \frac{0.08600}{0.00860} = 10.$$

4. A person deposited \$1,000 at interest for 10 years and received \$1,219 at the end of the 10 years. What rate of interest did he receive if interest was compounded annually?

$$\begin{aligned}
&P = 1{,}000, && S = P(1+r)^n, \\
&S = 1{,}219, && \log S = \log P + n\log(1+r), \\
&n = 10, && n\log(1+r) = \log S - \log P, \\
&r = ? && \log(1+r) = \frac{\log S - \log P}{n} \quad \text{[formula solved for } \log(1+r)]
\end{aligned}$$

$$\frac{\log S - \log P}{n} = \frac{0.08600}{10} = 0.00860,$$

$$\log(1+r) = 0.00860,$$

$$1 + r = 1.02,$$

$$r = 1.02 - 1 = 0.02 = 2\%.$$

---

85. Table of Compound Amount. We have seen that the formula for the amount at compound interest is

$$S = P(1+r)^n.$$

If $P = \$1$, this formula becomes

$$S = (1+r)^n. \tag{5}$$

If we compute the compound amount for \$1 at the given rate and for the given period of time, we may find the compound amount for any given sum by multiplying the compound amount for \$1 by that given sum. Thus, in Illustration 1 above, we may first compute $(1.02)^{10}$ as 1.218994 and then multiply by 1,000 to obtain 1,218.99, or \$1,219, as the amount to the nearest dollar.

To facilitate computations in compound interest problems, tables have been prepared which list the compound amount of \$1 for various rates and periods of time. Extensive tables of this type are labor-saving conveniences and are standard equipment for those who must perform many computations of this kind. Table III in the Appendix gives the compound amount of \$1 to six decimal places for values of n from 1 to 100 and for selected values of r from $\frac{5}{12}\%$ to 7%. A portion of this table is reproduced on the page opposite.

The letter n at the top of the first column refers to the number of periods of time (*conversion periods*) for which the compounding of interest occurs, and the per cents 1, 2, 3, 4, 5, 6, and 7 at the top of the other columns refer to the rate per conversion period. Thus, if it is stated that "the interest is 6% compounded *semiannually* for 2 years," there would be 4 conversion periods (4 six-month intervals), and the rate per period would be 3% (6% ÷ 2). To find the compound amount for $1, we would follow down the 3% column and read 1.125509 opposite $n = 4$. We must be careful, therefore, when the stated rate per year is given (nominal rate) to divide that rate by the number of conversion periods per year and to take as n the number of conversion periods.

AMOUNT OF $1 AT COMPOUND INTEREST (FROM TABLE III)

$$S = (1 + r)^n$$

n	1%	2%	3%	4%	5%	6%	7%
1	1.010000	1.020000	1.030000	1.040000	1.050000	1.060000	1.070000
2	1.020100	1.040400	1.060900	1.081600	1.102500	1.123600	1.144900
3	1.030301	1.061208	1.092727	1.124864	1.157625	1.191016	1.225043
4	1.040604	1.082432	1.125509	1.169859	1.215506	1.262477	1.310796
5	1.051010	1.104081	1.159274	1.216653	1.276282	1.338226	1.402552
6	1.061520	1.126162	1.194052	1.265319	1.340096	1.418519	1.500730
7	1.072135	1.148686	1.229874	1.315932	1.407100	1.503630	1.605781
8	1.082857	1.171659	1.266770	1.368569	1.477455	1.593848	1.718186
9	1.093685	1.195093	1.304773	1.423312	1.551328	1.689479	1.838459
10	1.104622	1.218994	1.343916	1.480244	1.628895	1.790848	1.967151
....	....	....	....	....	....	....	....
20	1.220190	1.485947	1.806111	2.191123	2.653298	3.207135	3.869684
....	....	....	....	....	....	....	....
50	1.644632	2.691588	4.383906	7.106683	11.467400	18.420154	29.457025
....	....	....	....	....	....	....	....
100	2.704814	7.244646	19.218632	50.504948	131.501258	339.302084	867.716326

Illustration: Find the compound amount of $200 at 4% compounded quarterly for 5 years. Here we have

$$n = 20 \qquad (5 \times 4 \text{ quarter periods}),$$
$$r = 0.01 \qquad (0.04 \div 4).$$

In the table we find for a rate of 1% and an n of 20, the compound amount for $1 to be 1.220190. Hence, for $200 we have for the required compound amount

$$200 \times 1.220190 = \$244.04.$$

PROBLEMS

1. Find the compound amount of \$460 for 4 years at 2% compounded semiannually.

Answer: \$498.11.

2. Find the compound amount for the following: (*a*) \$1,000 for 8 years at 1% compounded annually, (*b*) \$500 for 6 years at 2% compounded semiannually, (*c*) \$1,500 for 25 years at 4% compounded quarterly.

Answers:

(*a*) \$1,082.86, (*b*) \$563.41, (*c*) \$4,057.22.

86. Finding Time and Rate by Interpolation. The problem of finding the time (n) and the rate (r) per conversion period by the use of logarithms has been explained in Section 84. If a table of compound amount is used, the time or the rate may be approximated by interpolation as follows:

(*a*) *To Find Time*

$$S = P(1 + r)^n.$$

In what time will \$950 amount to \$1,000 at 2% compounded semiannually?

Substituting in the formula to find n,

$$1{,}000 = 950(1.01)^n,$$
$$(1.01)^n = 1.0526.$$

In the table in the 1% column we search for 1.0526. This value does not appear; but we find that it lies between 1.0510 below it for $n = 5$ and 1.0615 above it for $n = 6$. Then, if we assume that the change in $(1.01)^n$ is proportional to the change in n, we must determine where the given value 1.0526 lies between 1.0510 and 1.0615. The difference between these two values is 0.0105, and the difference between 1.0510 (the smaller value) and 1.0526 (the given value) is 0.0016. Hence, it follows by proportion that the required value of n is $\frac{16}{105}$ of the way from $n = 5$ to $n = 6$. This means that $n = 5 + \frac{16}{105}(1) = 5.152$, the required number of conversion periods. It follows that \$950 will amount to \$1,000 in 2.576 years, or approximately 2 years 6 months 27 days at 2% compounded semiannually. When logarithms are used, it turns out that $n = 5.158$, a slightly more accurate result.

(*b*) *To Find Rate*

$$S = (1 + r)^n.$$

At what rate, compounded annually, must a capital of \$1,400 be invested to amount to \$1,500 in 3 years?

Substituting in the formula to find r,

$$1{,}500 = 1{,}400(1 + r)^3,$$

$$(1 + r)^3 = 1.0714.$$

In the table opposite $n = 3$ we search for the value 1.0714 under the various rate headings. The nearest values found are 1.0612 less than it for $r = 2\%$ and 1.0927 greater than it for $r = 3\%$. Then, if we assume that the change in the compound amount for a small change in the interest rate is proportional to the change in the rate, an approximation to the required rate may be found by interpolation. In the usual way we reason that since 1.0714 is $\frac{102}{315}$ of the way between the two tabular values, the required rate is taken to be $\frac{102}{315}$ of the way from 2% to 3%. That is, $r = 0.02 + \frac{102}{315}(0.01) = 0.02 + 0.00324 = 0.02324$, or 2.324%. This approximation may be compared with 2.33% found by using logarithms.

PROBLEMS

1. How long will it take \$1,000 to amount to \$2,000 at 3% compound interest? Use interpolation, and check your answer by solving for n by use of logarithms.

Answer: 23.446 years; 23.445 years.

2. Mr. Brown deposits \$4,000 in a savings bank paying 4% per annum, compounded quarterly. He does not want his deposits in this bank to exceed \$5,000. Use interpolation and find out when this will happen, and check your answer by using logarithms.

Answer: 5.606 years; 5.608 years.

3. A United States savings bond is purchased for \$18.75 and is worth \$25 in 10 years. If interest is compounded semiannually, find the nominal interest rate by interpolation, and check your answer by use of the logarithmic method.

Answer: 2.894%; 2.90%.

4. If \$500, invested at interest compounded annually, amounts to \$800 in 10 years, find the interest rate by interpolation, and check by use of logarithms.

Answer: 4.806%; 4.81%.

Exercises

1. Complete the following list of compound interest problems:

	Principal	*Amount*	*Rate*	*Time*
(*a*)	\$100	\$_____	$1\frac{1}{2}\%$	6 years
(*b*)	500	609.50	2%	_____
(*c*)	_____	317.05	2%	12 years
(*d*)	800	996.96	_____	5 years

2. The sum of \$350 was deposited in a savings bank which paid interest at the rate of $1\frac{1}{2}\%$ per year. Find the compound amount for a period of 20 years. How much interest was earned?

3. Accumulate \$1,000 for 9 years at $1\frac{1}{4}\%$ compounded annually.

4. Compare with the compound interest on \$1,600 for 10 years at 3.5% compounded annually the simple interest on the same sum for 10 years at the same rate.

5. Find the compound amount of \$5,000 for 3 years 6 months at 2%, compounded quarterly. (*Hint:* There would be 14 conversion periods at a rate of $\frac{1}{2}\%$ per quarter.)

6. A man wished to set aside a certain sum of money as an educational fund for his young son. How much must be set aside in a savings bank paying 1% per year in order that there will be \$3,000 available for college expenses 18 years later? (Use logarithms.)

In Problems 7, 8, and 9, compare answers obtained by the use of logarithms with those obtained by the use of the table of compound amount.

7. If \$100 is invested to yield $1\frac{1}{4}\%$ compound interest and accumulates until it has amounted to \$105.09, what is the time of the investment?

8. How long will it take a sum of money to double itself at 2% compound interest?

9. The sum of \$200 accumulated at compound interest for 8 years, at which time it amounted to \$238.96. What was the rate of interest?

10. At what rate will any sum double itself in 5 years? 10 years? 30 years?

11. Find the present value of \$400 due in 5 years if money is worth 3% compounded annually. (*Hint:* Find P in the formula for compound amount.)

12. Find the compound amount in each of the following:

	Principal	*Rate*	*Compounded*	*Time*
(*a*)	\$ 700	$2\frac{1}{2}\%$	Semiannually	25 years
(*b*)	1,100	2%	Quarterly	5 years
(*c*)	300	$1\frac{1}{2}\%$	Annually	10 years
(*d*)	50	6%	Monthly	4 years

13. Which is worth more now, \$1,000 due in 3 years or \$1,100 due in 5 years, if money is worth 3% compounded annually?

14. Use interpolation in Table III, and supply the missing quantities below:

	Principal	*Amount*	*Rate*	*Time*
(*a*)	\$ 1,000	\$ 1,175.00	———	19 years
(*b*)	10,000	14,425.00	———	31 years
(*c*)	100	141.30	2%	———
(*d*)	25	50.00	6%	———

87. Present Value. We have seen that the compound amount of a sum of money is given by the formula

$$S = P(1 + r)^n,$$

in which S = amount, P = principal, r = rate per period, and n = number of periods. Thus, if \$1,000 is placed at interest for 5 years at 4% compounded annually, the amount is found to be

$$S = 1{,}000(1.04)^5 = \$1{,}216 \text{ (approximately)}.$$

Suppose that it were required to have \$1,216 available at the end of 5 years, what principal would have to be placed at 4% interest compounded annually? The answer obviously is \$1,000. However, this principal could have been determined by solving for P in the equation

$$\$1{,}216 = P(1.04)^5.$$

We have $P = \dfrac{1{,}216}{(1.04)^5} = 1{,}216 \times (1.04)^{-5}$

$$= 1{,}216 \times \frac{1}{1.216653}.$$

$$P = 1{,}216 \times 0.821927 = \$1{,}000 \text{ (approximately)}.$$

The principal which, invested at interest, will accumulate to a specified sum at a future date is known as the present value of that sum. Thus, as illustrated above, the present value of an amount S, due in n periods, when money can be invested at rate r is

$$P = \frac{S}{(1 + r)^n} = S(1 + r)^{-n}. \tag{6}$$

Table IV (page 389) gives the present value of \$1 at compound interest for certain values of r and n. A portion of the table follows.

Present Value of \$1 at Compound Interest (from Table IV)

$(1 + r)^{-n}$

n	1%	2%	3%	4%	5%	6%	7%
1	0.990099	0.980392	0.970874	0.961538	0.952381	0.943396	0.934579
2	0.980296	0.961169	0.942596	0.924556	0.907029	0.889996	0.873439
3	0.970590	0.942322	0.915142	0.888996	0.863838	0.839619	0.816298
4	0.960980	0.923845	0.888487	0.854804	0.822702	0.792094	0.762895
5	0.951466	0.905731	0.862609	0.821927	0.783526	0.747258	0.712986
6	0.942045	0.887971	0.837484	0.790315	0.746215	0.704961	0.666342
7	0.932718	0.870560	0.813092	0.759918	0.710681	0.665057	0.622750
8	0.923483	0.853490	0.789409	0.730690	0.676839	0.627412	0.582009
9	0.914340	0.836755	0.766417	0.702587	0.644609	0.591898	0.543934
10	0.905287	0.820348	0.744094	0.675564	0.613913	0.558395	0.508349
......	....	....	....	....	....	....	....
20	0.819544	0.672971	0.553676	0.456387	0.376889	0.311805	0.258419
......	....	....	....	....	....	....	....
50	0.608039	0.371528	0.228107	0.140713	0.087204	0.054288	0.033945
......	....	....	....	....	....	....	....
100	0.369711	0.138033	0.052033	0.019800	0.007604	0.002947	0.001152

In the example above, we find 0.821927 in the 4% column opposite an n of 5 and multiply this value by \$1,216 to obtain \$1,000 as the present value of the given amount.

Interpolation may be used to find approximate values for r and n not appearing in the present value table in the same way that it was used in the case of the compound amount table.

Illustration: Find the present value of \$2,400 due in 5 years if the rate of interest is 4% compounded semiannually?

Solution: $P = S(1 + r)^{-n}$, in which $S = \$2{,}400$, $n = 10$ (2 periods per year), and $r = 0.02$ (since 4% annual rate equals 2% per semiannual period). In the 2% column of the table opposite an n of 10 we read 0.820348. Hence, $P = \$2{,}400 \times 0.820348 = \$1{,}968.84$.

Problems

1. Find the present value of (*a*) \$400 due in 3 years at 3%, (*b*) \$600 due in 2 years at 3%, (*c*) \$800 due in 1 year at 3%.

2. Find the present value of \$1,000 due in 3 years if the rate of interest is 4% compounded (*a*) annually, (*b*) semiannually, (*c*) quarterly.

3. What amount of money invested now at 3%, compounded semiannually, will amount to \$1,000 in 5 years? In 10 years?

4. What sum placed at interest now will amount to \$500 in 5 years if it earns 3% compounded annually? Quarterly? Bimonthly?

5. Finding the present value of a sum due in n years is sometimes referred to as discounting the sum. Discount \$10,000 for 10 years if money is worth 2%; 3%; 4%.

6. The difference between the sum discounted and its present value is sometimes called *compound discount.* Find the compound discount if \$1,000 is discounted for 5 years, money worth 3%; for 7 years; for 10 years.

7. Evaluate $(1.015)^{-10}$; $(1.07)^{-7}$; $1/(1.005)^5$.

8. Recall from Section 57 that the present value P of an amount A for the simple interest case is given by the formula

$$P = \frac{A}{1 + rt},$$

where r is the simple interest rate per year and t is the number of years. Compare the present values at simple and compound interest for an amount of \$100 with $r = 5\%$ and $t = 5$ years.

9. Show that $1/(1 + r)^3$ is less than $1/(1 + 3r)$ and hence that the present value of a sum due in 3 years is less if interest is compound than if it is simple.

10. Smith may pay \$300 cash for a refrigerator or \$305 at the end of 6 months. If money is worth 3% compounded semiannually, which is the better deal, and how much better?

88. Effective Rate. A dollar compounded semiannually for a year at 5% amounts to $\left(1 + \frac{0.05}{2}\right)^2$, or $(1.025)^2$, which equals 1.050625. If the same dollar had been compounded annually at the rate 0.050625, it would have also amounted to $1 + 0.050625$, or 1.050625. Hence, we see that the rate 0.050625 on an annual basis (effective rate) is equivalent to the rate 0.05 (nominal rate) on a semiannual basis.

In general, if i stands for effective rate, j for nominal rate, and m for the number of conversion periods per year, we have the relation

$$1 + i = \left(1 + \frac{j}{m}\right)^m, \tag{7}$$

which gives the effective rate in terms of the equivalent nominal rate.

Illustrations:

1. Find the effective rate equivalent to 2% compounded quarterly.

We have $i =$ effective rate,

$$j = 0.02 \qquad \text{(nominal rate)},$$

$$m = 4.$$

Hence, $$1 + i = \left(1 + \frac{0.02}{4}\right)^4,$$

$$= (1.005)^4 = 1.020151,$$

and $$i = 1.020151 - 1 = 0.020151.$$

2. Find the nominal rate compounded monthly that is equivalent to 2% effective.

We have $j =$ nominal rate,

$$i = 0.02 \qquad \text{(effective rate)},$$

$$m = 12.$$

Hence, $$1 + 0.02 = \left(1 + \frac{j}{12}\right)^{12},$$

which, when solved for j, yields (using 7-place tables) $j = 0.0198$.

PROBLEMS

1. Find the effective rate equivalent to (*a*) 2% compounded semiannually, (*b*) 2% compounded quarterly, (*c*) 4% compounded monthly, (*d*) k% compounded quarterly.

2. Find the nominal rate compounded semiannually equivalent to (*a*) 2% effective, (*b*) 1% effective, (*c*) 4% effective, (*d*) k% effective.

3. Find the nominal rate compounded monthly equivalent to $k\%$ effective.

4. Find the nominal rate compounded quarterly that is equivalent to a nominal rate of 2.9% compounded semiannually.

5. Find the effective interest rate paid on a savings account that pays nominal interest of $2\frac{1}{2}\%$ compounded quarterly.

89. Amount of Ordinary Annuity. A man deposits $1,000 at the end of each year in a savings bank paying 2%. How much will he have on deposit at the end of 5 years?

Since the first deposit is made at the end of the first year, it will be at interest for 4 years and will, by formula (4) of Section 84, amount to $\$1{,}000(1 + 0.02)^4$. The second deposit will be at interest for 3 years and will amount to $\$1{,}000(1 + 0.02)^3$; the third will amount to $\$1{,}000(1 + 0.02)^2$; and the fourth to $\$1{,}000(1 + 0.02)^1$. The final deposit is made at the end of the fifth year and will consequently be $1,000. The total amount to which these periodic deposits accumulate will be the amount on deposit at the end of 5 years. The sum may be written

$$\$1{,}000 + \$1{,}000(1 + 0.02)^1 + \$1{,}000(1 + 0.02)^2 + \$1{,}000(1 + 0.02)^3 + \$1{,}000(1 + 0.02)^4,$$

or
$$\$1{,}000 + \$1{,}020 + \$1{,}040.40 + \$1{,}061.21 + \$1{,}082.43 = \$5{,}204.04.$$

A series of equal periodic payments is called an annuity, and the sum to which the payments accumulate at compound interest is known as the amount of the annuity. Equal periodic payments on a piece of property, monthly payments of rent, insurance beneficiary payments, and so forth, are some of the examples of annuities encountered in the business world.

A thorough study of the subject of annuities would call for a detailed consideration of several types classified according to length of term, time of payment, and so forth. In this book we shall confine our attention chiefly to the type known as *ordinary annuity,* in which payments are made at the *end* of each period, and, unless otherwise stated, the word *annuity* will here mean an ordinary annuity.

If each payment is R dollars and continues for n years, there will be n payments. The first payment will be at interest for $n - 1$

years and will amount to $R(1+r)^{n-1}$. The second payment will be at interest for $n-2$ years and will amount to $R(1+r)^{n-2}$. The next to the last payment will be at interest for 1 year and will amount to $R(1+r)^1$. The final payment, made at the end of the last year, will consequently be R. The total amount of the annuity will be the sum of the amounts of the separate payments. Writing these in reverse order, we have for the sum

$$S = R + R(1+r) + R(1+r)^2 + \cdots + R(1+r)^{n-2} + R(1+r)^{n-1},$$

or

$$S = R[1 + (1+r) + (1+r)^2 + \cdots + (1+r)^{n-2} + (1+r)^{n-1}].$$

The expression within brackets is a geometric series of n terms and with common ratio $1+r$. Its sum s, by the formula for the sum of a geometric series, is

$$s = \frac{(1+r)^n - 1}{(1+r) - 1}, \quad \text{or} \quad s = \frac{(1+r)^n - 1}{r}. \tag{8}$$

Hence,

$$S = R\,\frac{(1+r)^n - 1}{r}. \tag{9}$$

Formula (8) gives the amount of an annuity on the basis of payments of \$1 each; formula (9) shows that if each payment is R dollars, instead of \$1, the accumulation will be R times as much.

To illustrate the use of formula (9) we return to the problem at the beginning of this section. We have $R = \$1{,}000$, $r = 0.02$, $n = 5$. Hence,

$$S = \$1{,}000\,\frac{(1.02)^5 - 1}{0.02} = \$1{,}000\,\frac{1.10408 - 1}{0.02}$$

$$= \$1{,}000 \times \frac{0.10408}{0.02} = \$5{,}204 \text{ (approximately).}$$

Table V (page 397) gives the values of the amount of an annuity of \$1 for certain values of r and n and may be used to advantage in finding the amount of an annuity. The process consists simply in locating the amount for \$1 under the proper per cent column (r) and opposite the proper n. In our example, we find 5.204040 in the 2% column opposite an n of 5 and multiply this value by \$1,000 to obtain \$5,204.04 as the amount of the annuity required. A portion of the table follows:

AMOUNT OF AN ANNUITY (FROM TABLE V)

$$\frac{(1 + r)^n - 1}{r}$$

n	1%	2%	3%	4%	5%	6%	7%
1	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.00000
2	2.010000	2.020000	2.030000	2.040000	2.050000	2.060000	2.07000
3	3.030100	3.060400	3.090900	3.121600	3.152500	3.183600	3.21490
4	4.060401	4.121608	4.183627	4.246464	4.310125	4.374616	4.43994
5	5.101005	5.204040	5.309136	5.416323	5.525631	5.637093	5.75073
6	6.152015	6.308121	6.468410	6.632975	6.801913	6.975319	7.15329
7	7.213535	7.434283	7.662462	7.898294	8.142008	8.393838	8.65402
8	8.285671	8.582969	8.892336	9.214226	9.549109	9.897468	10.25980
9	9.368527	9.754628	10.159106	10.582795	11.026564	11.491316	11.97798
10	10.462213	10.949721	11.463879	12.006107	12.577893	13.180795	13.81644
..	....	....	....	....	....	....	....
20	22.019004	24.297370	26.870374	29.778079	33.065954	36.785591	40.99549
..	....	....	....	....	....	....	....
50	64.463182	84.579401	112.796867	152.667084	209.347996	290.335905	406.52892

Interpolation may be used to find approximate values for n and r when their corresponding tabular values do not appear in the table. Ordinarily, logarithms should be used when greater accuracy is desired.

Illustration 1: At the end of each 6 months a man deposits \$200 in a fund that pays interest at the rate of 4% per year compounded semiannually. What sum of money does he have to his credit at the time he has made the 8th deposit?

Solution: $R = 200,\ n = 8,\ r = 0.02.$

Substituting in formula (9) to find S, we obtain

$$S = 200\,\frac{(1.02)^8 - 1}{0.02}.$$

Using Table V with the given entries, we have

$$S = 200(8.582969), \text{ or } \$1{,}716.59,$$

the amount to his credit immediately after making the 8th deposit.

Illustration 2: A father started a savings account for his son by depositing \$100 to the son's credit when the son was 1 year old, in a bank that paid $2\frac{1}{2}\%$ compounded annually. If the payments were kept up each year until the son's 21st birthday, how much was then to the son's credit?

Solution: $R = 100,\ n = 21,\ r = 0.025.$

Substituting in formula (9) to find S, we obtain

$$S = 100\,\frac{(1.025)^{21} - 1}{0.025}.$$

Since the given rate is not included in Table V, we shall use logarithms to complete the solution. The steps are as follows:

Find $(1.025)^{21}$ by logarithms:

$$\begin{aligned} \log 1.025 &= 0.01072, \\ 21 \log 1.025 &= 0.22512, \\ (1.025)^{21} &= 1.6793. \end{aligned}$$

Our expression for S may now be written.

$$S = 100\,\frac{1.6793 - 1}{0.025} = 100\,\frac{0.6793}{0.0250} = \frac{67.93}{0.025}.$$

$$\begin{aligned} \log S = \log 67.93 - \log 0.025 = 1.83206 - (8.39794 - 10)& \\ = 3.43412.& \end{aligned}$$

Hence, S = \$2,717.20, the amount to the son's credit on his 21st birthday.

PROBLEMS

1. A firm deposits \$2,400 a year for 10 years for establishing a trust fund. If the fund earns 1% a year, find the amount of the fund.

2. Find the value of an annuity of \$200 at the end of each year for 15 years if invested at 2% compounded annually.

3. Solve formula (9) for R.

4. A man has a debt of \$4,000 due in 5 years. In order to pay off this debt, he deposits a certain sum, at the end of each year for

the 5 years, in a fund which bears 3% interest. How much must his annual payment be?

5. Find the amount of an annuity of $80 at the end of each month for 8 years. Assume money is worth 5% compounded monthly.

6. If money is worth 6% compounded semiannually, what single payment at the end of 5 years is equivalent to $100 paid semiannually for 5 years?

7. Mr. Brown saves $200 every three months and puts it in a savings bank paying 2% compounded quarterly. Find his total savings at the end of 10 years.

8. A bank has a Christmas savings plan whereby the customer deposits $10 at the end of each month starting with January and collects his savings at the end of November. If the bank pays 5% compounded monthly, what is the amount of the Christmas fund?

9. To provide for the availability of a sum of money at the end of 17 years, a firm places $1,000 at the end of each year in a sinking fund paying 4%. What sum will be available at the end of the period?

10. The Jones family wants to deposit enough money every 3 months in a college fund for their new son so that $3,000 will be available at the end of 18 years for his college expenses. If their money earns 3% compounded quarterly, what quarterly deposit should be made?

11. To meet an obligation of $100,000 at the end of 20 years, what amount of money should a firm put into a fund at the end of each year if the fund draws interest at 2%?

12. How long will it take to accumulate $1,000 by quarterly deposits of $20 each if the deposits earn 2% compounded quarterly?

13. How many monthly payments of $10 each are necessary to amount to $1,000 if the payments draw interest at 5% compounded monthly?

90. Present Value of Ordinary Annuity. In September, prior to his son's graduation from high school, a father purchases an annuity which will pay his son $1,000 at the end of each year for the next 4 years. If the trust company pays 2% per year compounded annually, what is the cost of the annuity?

The first payment of \$1,000 is received 1 year after the "present" date of the annuity, or, in this case, the date of purchase. This payment, therefore, has a present value of $\$1{,}000(1 + 9.02)^{-1}$. The second payment is received 2 years after the "present" date and has, therefore, a present value of $\$1{,}000(1 + 0.02)^{-2}$, and so on, for the other two payments.

The "present value" of all the payments is seen to be the sum of the present values of the separate payments, that is,

$$\begin{aligned}P &= \$1{,}000[(1 + 0.02)^{-1} + (1 + 0.02)^{-2} + (1 + 0.02)^{-3} \\ &\qquad + (1 + 0.02)^{-4}] \\ &= \$1{,}000(0.980392 + 0.961169 + 0.942322 + 0.923845) \\ &= \$1{,}000(3.807728) = \$3{,}807.73, \text{ cost of the annuity.}\end{aligned}$$

Since the present value of the first payment of an annuity of \$1 is $(1 + r)^{-1}$ and that of the second payment is $(1 + r)^{-2}$, and so on, and the present value of the last payment made at the end of n years is $(1 + r)^{-n}$, we may obtain a formula for the sum of the present values of all the payments in terms of r and n. This sum is called the *present value of an annuity.*

By application of the formula for the sum of the terms of a geometric series, we have for the sum p of the present values of all of the \$1 annuity payments

$$\begin{aligned}p &= (1 + r)^{-1}\frac{1 - (1 + r)^{-n}}{1 - (1 + r)^{-1}} \\ &= \frac{1}{1 + r}\cdot\frac{1 - (1 + r)^{-n}}{1 - (1 + r)^{-1}} \\ &= \frac{1 - (1 + r)^{-n}}{r}.\end{aligned}$$

If the annual payment is R instead of \$1, the present value of the annuity is

$$P = R\,\frac{1 - (1 + r)^{-n}}{r}. \tag{10}$$

Illustration:

Find the present value of an annuity of \$500 a year for 6 years at 4%.

$$P = R\,\frac{1-(1+r)^{-n}}{r}$$

$$= \$500\,\frac{1-(1.04)^{-6}}{0.04}$$

$$= \$500\,\frac{1-0.790315}{0.04} = \$500 \times 5.2421$$

$$= \$2{,}621 \text{ (approximately).}$$

Table VI (page 405) gives the present value of an annuity of $1 per period for certain values of r and n. A portion of the table is shown below.

To use the present value table in Illustration 1 instead of the formula, we read 5.242137 in the 4% column opposite an n of 6 and multiply this value by $500. We obtain $500 × 5.242137 = $2,621 (approximately) as before.

PRESENT VALUE OF ANNUITY (FROM TABLE VI)

$$\frac{1-(1+r)^{-n}}{r}$$

n	1%	2%	3%	4%	5%	6%	7%
1	0.990099	0.980392	0.970874	0.961538	0.952381	0.943396	0.934579
2	1.970395	1.941561	1.913470	1.886095	1.859410	1.833393	1.808018
3	2.940985	2.883883	2.828611	2.775091	2.723248	2.673012	2.624316
4	3.901966	3.807729	3.717098	3.629895	3.545951	3.465106	3.387211
5	4.853431	4.713460	4.579707	4.451822	4.329477	4.212364	4.100197
6	5.795476	5.601431	5.417191	5.242137	5.075692	4.917324	4.766540
7	6.728194	6.471991	6.230283	6.002055	5.786373	5.582381	5.389289
8	7.651678	7.325481	7.019692	6.732745	6.463213	6.209794	5.971299
9	8.566018	8.162237	7.786109	7.435332	7.107822	6.801692	6.515232
10	9.471305	8.982585	8.530203	8.110896	7.721735	7.360087	7.023582
.....	....	....	....	....	....	....	....
20	18.045553	16.351433	14.877475	13.590326	12.462210	11.469921	10.594014
.....	....	....	....	....	....	....	....
50	39.196118	31.423606	25.729764	21.482185	18.255925	15.761861	13.800746
.....	....	....	....	....	....	....	....
100	63.028879	43.098352	31.598905	24.504999	19.847910	16.617546	14.269251

PROBLEMS

1. Find the present value of an annuity of $50 a year for 4 years at 2%.

2. Find the present value of an annuity of $1,200 a year for 10 years at 3%.

3. A man is offered a house for $12,000 cash or for $2,000 down and payments of $1,200 a year for 10 years. If money is worth 3%, which is the better deal for the purchaser, and by how much?

4. What is the cash value now of an annuity of $100 a month for the next 8 years if money is worth 5% compounded monthly?

5. Find the sum of money needed to set up a fund paying $100 a quarter for 10 years if the money earns 3% compounded quarterly.

6. What is the cost of an annuity of $500 per quarter for 8 years if money can earn 4% compounded quarterly?

7. Mr. J. R. Doe buys a house for $4,500 cash and $199.74 a quarter for 20 years. Find the selling price of the house if money is worth $4\frac{1}{2}$% compounded quarterly.

8. A finance company advertised that it would loan $1,000 to be paid back in 20 monthly payments of $58.70. Find the present value of this annuity if money is assumed to be worth 18% compounded monthly. Is the company earning more or less than 18% compounded monthly?

9. A loan company advertises that it will loan $100 to be paid back in 15 monthly payments of $8.40. By interpolation in Table VI, find the approximate rate, compounded monthly, which the loan company assumes money to be worth.

10. Smith has the choice of paying $1,000 cash for a lot or of making bimonthly payments of $100 for 2 years. Which is the better offer, and how much better, if Smith can invest his money at 6% compounded bimonthly?

91. Amortization and Sinking Funds. A student plans to borrow $1,000 to meet part of his college expenses and is offered his choice of the following two methods for repaying the loan:

(*a*) He may make 5 equal annual payments, which are to include interest on the debt at the time of payment and a sum which reduces the principal. The interest is to be 4% compounded annually.

(*b*) He may make 5 equal deposits into a fund which on maturity will be sufficient to repay the loan. In addition, he is to pay his creditor 5% simple interest per year on the amount of the loan; and the fund is to earn for the student 4% per year compounded annually.

Find the annual expense required to repay the loan by each of these two methods.

(*a*) *Amortization Method.* The \$1,000 may be looked upon as the present value of a series of 5 payments with money worth 4% compounded annually. Hence, if we take P = \$1,000, $n = 5$, and $r = 0.04$, the value of R in formula (10) will be the required annual payment by this method.

Using Table VI, we read 4.451822 in the 4% column opposite $n = 5$. We then have

$$\$1{,}000 = R \times 4.451822,$$

from which $R = \$1{,}000 \div 4.451822 = \$224.63$, annual payment.

Hence, \$224.63 at the end of each year will repay the loan, including principal and interest.

By this method, it is seen that each payment includes the interest on the unpaid balance of the debt and at the same time reduces the amount of the loan. This method of reducing a debt is called *amortization,* and a schedule that shows the division of the payments into interest and debt reduction is called an *amortization schedule.*

Amortization Schedule

Period (One Year)	*Interest Due on Debt Outstanding*	*Payment (Annuity)*	*Reduction of Debt*	*Debt Outstanding*
Date of loan.........	....	....	....	\$1,000.00
End of 1st year......	40.00	224.63	184.63	815.37
End of 2nd year.....	32.61	224.63	192.02	623.35
End of 3rd year......	24.93	224.63	199.70	423.65
End of 4th year......	16.95	224.63	207.68	215.97
End of 5th year......	8.64	224.63	215.99	
Total.............	\$123.13	\$1,123.15	\$1,000.02	

The slight discrepancy (\$0.02) in the above schedule is due to rounding off annual payments to the nearest cent. A check on the accuracy of the computations may be made since the total of the Interest column plus the original debt should equal the total of the Payment, or Annuity, column.

The construction of an amortization schedule when payments are made semiannually, quarterly, and so on, is similar except that the period used must be taken accordingly.

(*b*) *Sinking Fund Method.* According to the second choice offered as a method of repaying the loan, the annual payment consists of two parts—the annuity payment, which will amount to $1,000 in 5 years, plus the annual interest on the loan.

Substituting $S = \$1{,}000$, $n = 5$, and $r = 0.04$ in formula (9), we may find the annual annuity payment by solving for R. Alternatively, using Table V, we read 5.416323 in the 4% column opposite $n = 5$. We then have

$$\$1{,}000 = R \times 5.416323,$$

from which $R = \$1{,}000 \div 5.416323 = \$184.63$, annuity payment.

In addition, the interest per year on the loan is $0.05 \times \$1{,}000 = \$50$. This means that the total annual payment required to repay the loan is $184.63 + $50, or $234.63.

This method, which consists essentially in finding the annuity which will amount to the given debt, is known as the *sinking fund* method of repaying a debt. The operation of this method can be observed by constructing a *sinking fund schedule.* The difference between the total amount of the original debt and the amount in the sinking fund at any particular time is known as the ***book value*** of the debt at that time.

SINKING FUND SCHEDULE

Period (One Year)	*Total Payment*	*Sinking Fund Charge*	*Interest Earned*	*Contribution to Sinking Fund*	*Amount in Sinking Fund*	*Book Value of Debt*
Date of loan.....	....	....	....	....	....	$1,000.00
End of 1st year...	$ 234.63	$184.63	....	$ 184.63	$ 184.63	815.37
End of 2nd year..	234.63	184.63	$ 7.39	192.02	376.65	623.35
End of 3rd year...	234.63	184.63	15.07	199.70	576.35	423.65
End of 4th year...	234.63	184.63	23.05	207.68	784.03	215.97
End of 5th year...	234.63	184.63	31.36	215.99	1,000.02	
Total.........	$1,173.15	$ 923.15	$76.87	$1,000.02		

Checks upon the computations in the schedule may be made by observing that the total of the Sinking Fund Charges plus the total of the Interest Earned should equal the debt and that the total of the Contribution to Sinking Fund column should also equal the debt. The slight discrepancy ($0.02) in the above schedule is due to rounding off calculations to the nearest cent.

Finding the amount in a sinking fund at any given time is equivalent to finding the amount of an annuity for the appropriate annual payment, period, and rate. For example, we may find $376.65, the amount in the sinking fund at the end of the second year, as follows:

Substitute $R = \$184.63$, $n = 2$, $r = 0.04$ in formula (9), and solve for S; or, alternatively, use Table V, and read 2.040000 opposite $n = 2$ in the 4% column, obtaining finally

$$S = \$184.63 \times 2.040000 = \$376.65,$$

which agrees with the amount scheduled.

PROBLEMS

1. Smith is to discharge a debt of $2,000, bearing interest at 5% payable annually, by making equal annual payments for 5 years. Find the annual payment, and construct an amortization schedule.

2. Mr. Brown is to discharge a debt of $1,000, bearing interest at 6% payable semiannually, by making equal semiannual payments for 3 years. Find the semiannual payment, and construct an amortization schedule.

3. A loan of $5,000 with interest at 5% payable quarterly is to be paid off in equal quarterly installments for $1\frac{1}{2}$ years. What is the quarterly payment? Construct an amortization schedule.

4. A student borrowed $1,000 to meet part of his college expenses. He agreed to repay the loan with interest at 2% compounded semiannually by making semiannual payments over a period of 10 years. What payment must be made at the end of each 6-month period? Find the debt outstanding just after the payment is made at the end of the 5th year. (*Note:* The debt outstanding at any time is the present value of the payments still to be made.)

5. A man is offered a house for $12,000. He pays for it with $2,000 of his own funds and $10,000 borrowed from a bank. He is

to discharge the debt by equal semiannual payments over a period of 15 years. If the debt bears interest at 6% payable semiannually, what is the amount of each payment? Construct an amortization schedule for the first 2 years.

6. Work Problem 1 if the debt is to be discharged by the sinking fund method where deposits in the sinking fund earn interest at 4% compounded annually.

7. What is the book value of the debt in Problem 6 after the 3d payment?

8. A debt of $10,000 with interest at 5% payable annually is due in 10 years. If the debt is to be discharged by the sinking fund method, find the annual expense if the sinking fund earns interest at 4% compounded annually. Construct a sinking fund schedule for the first 4 years.

9. What is the book value of the debt in Problem 8 after the 7th payment?

10. A school district borrows $50,000 and is to pay $750 interest every 6 months until the debt is discharged. If the debt is to be paid by contributions to a sinking fund every 6 months for 20 years and if the fund pays interest at 3% compounded semiannually, find the semiannual expense. How much is in the sinking fund at the end of the 5th year?

92. The Binomial Formula. Let us return to the example with which we began Section 84 of this chapter. We saw there that the compound amount of $100 at 3% for 5 years was given by the expression

$$S = \$100(1 + 0.03)^5.$$

The quantity $(1 + 0.03)$ is said to be a binomial, and the quantity $(1 + 0.03)^5$ is said to be "The binomial $(1 + 0.03)$ raised to the fifth power." In general, any expression containing two terms is called a binomial, whether these terms be numbers or letters or both. Thus, $(a + b)$, $(2x - y)$, $(a^2 - 3)$, $(1 + i)$, are all examples of binomials.

Now, the value of the expression $(1 + 0.03)^5$ could readily be found by multiplying 1.03 by itself 4 times, thus:

$$(1 + 0.03)^5 = 1.03 \times 1.03 \times 1.03 \times 1.03 \times 1.03 = 1.1592740743.$$

But there are many times when it is advantageous to employ a general formula for finding the value of a binomial expression. We shall consider the form $(a + b)^n$ and proceed to show how such a formula may be found if n is a positive integer.

By actual multiplication, we obtain the following products:

$$(a + b)^2 = a^2 + 2ab + b^2,$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4,$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

For example, for the first product the multiplication may be carried out as follows:

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{ + ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

There are certain properties that may be noticed in any one of the above *expansions*. The power of the first term is always the same as that of the binomial. The exponents of a decrease by 1 and those of b increase by 1 from term to term. The sum of the exponents for any term is equal to n, the exponent of the binomial. If the coefficient in any one term is multiplied by the exponent of a and this product is divided by one more than the exponent of b, the result is the coefficient for the next term. The arrangement of the coefficients is symmetrical in that they are the same at corresponding positions from the ends of the expansion. Furthermore, there are $(n + 1)$ terms in each expansion.

PROBLEMS

1. Write the expansion of $(a + b)^6$ by making use of the properties pointed out above.

Answer: $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.

If we assume that these properties hold for *all* positive integral values of $n (n = 1, 2, 3, 4, \ldots,$ and so forth), we may bring our findings together in one formula, known as the *binomial formula.*

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \times 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3}b^3 + \cdots + nab^{n-1} + b^n.$$

Using the binomial formula to compute the value of $(1 + 0.03)^5$, we verify, by substituting, that

$$
\begin{aligned}
(1 + 0.03)^5 &= (1)^5 + 5(1)^4(0.03) + \frac{(5)(4)}{1 \times 2}(1)^3(0.03)^2 \\
&\quad + \frac{(5)(4)(3)}{1 \times 2 \times 3}(1)^2(0.03)^3 + \frac{(5)(4)(3)(2)}{1 \times 2 \times 3 \times 4}(1)(0.03)^4 + (0.03)^5 \\
&= (1)^5 + 5(1)^4(0.03) + 10(1)^3(0.03)^2 + 10(1)^2(0.03)^3 \\
&\qquad + 5(1)(0.03)^4 + (0.03)^5 \\
&= 1 + 0.15 + 0.009 + 0.00027 + 0.00000405 \\
&\qquad + 0.0000000243 \\
&= 1.1592740743, \text{ or } 1.1593, \text{ rounding off to four decimal places.}
\end{aligned}
$$

2. Find 4 terms of the expansion of $(1 + 0.06)^9$.
Answer: $1 + 9(.06) + 36(.06)^2 + 84(.06)^3$.

If the same laws that apply to the formation of the successive terms of the binomial formula are used in the expansion of the binomial $(a + b)^n$, where n is any real number *not* a positive integer, nor zero, the series of terms that results is unending. For example, if $n = \frac{1}{2}$, the binomial expression becomes

$$
\begin{aligned}
(a + b)^{\frac{1}{2}} &= a^{\frac{1}{2}} + \left(\frac{1}{2}\right)a^{\frac{1}{2}-1}b + \frac{(\frac{1}{2})(\frac{1}{2} - 1)}{2!}a^{\frac{1}{2}-2}b^2 \\
&\qquad + \frac{(\frac{1}{2})(\frac{1}{2} - 1)(\frac{1}{2} - 2)}{3!}a^{\frac{1}{2}-3}b^3 + \cdots \text{ and so forth} \\
&= a^{\frac{1}{2}} + \left(\frac{1}{2}\right)a^{-\frac{1}{2}}b + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}a^{-\frac{3}{2}}b^2 \\
&\qquad + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}a^{-\frac{5}{2}}b^3 + \cdots \text{ and so forth} \\
&= a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8}a^{-\frac{3}{2}}b^2 + \tfrac{3}{48}a^{-\frac{5}{2}}b^3 + \cdots \text{ and so forth.}
\end{aligned}
$$

We shall assume the binomial formula to hold true for any real n, provided that the numerical value of b/a is less than 1.

Illustration: Find the approximate square root of 37 by using 2 terms of the binomial formula.

$$\sqrt{37} = (36 + 1)^{\frac{1}{2}}.$$

Here, $a = 36$, and $b = 1$. Substituting in the expansion for $(a + b)^{\frac{1}{2}}$, we have:

$$\begin{aligned}\sqrt{37} &= (36 + 1)^{\frac{1}{2}} = 36^{\frac{1}{2}} + (\tfrac{1}{2})36^{-\frac{1}{2}}(1) + \cdots \\ &= 6 + \tfrac{1}{12} = 6.08 \text{ (approximately).}\end{aligned}$$

Problems

1. Find the approximate cube root of 126 by using 3 terms of the binomial expansion of $(125 + 1)^{\frac{1}{3}}$.

2. Approximate the square roots of the following by using 3 terms of the binomial series: (*a*) 99, (*b*) 17, (*c*) 1.1, (*d*) 1.025, (*e*) 15, (*f*) 102.

3. Approximate, by using 3 terms of the binomial formula, the 5th root of (*a*) 1.01, (*b*) 1.1.

4. Find the first four terms in the expansion of:

(*a*) $(1 + x)^{-2}$,
(*b*) $(1 + x)^{-1}$,
(*c*) $(1 + r)^{-\frac{1}{2}}$,
(*d*) $(1 + 2x)^{\frac{1}{3}}$,
(*e*) $(1 - R)^{-2}$,
(*f*) $(1 - P)^{-1}$.

5. Evaluate $(1 + x)^{-2}$ when $x = \frac{1}{2}$, and compare the true value with the approximation obtained by using the first 4 terms of the expansion.

6. Substitute the binomial expansion of $(1 + r)^n$ into formula (8), and obtain the following formula for the sum of an annuity of \$1:

$$s = n + \frac{n(n-1)}{2}r + \frac{n(n-1)(n-2)}{2 \times 3}r^2 + \cdots.$$

7. Use the first 3 terms of the formula of Problem 6, and find the amount of an annuity of \$1 for 8 years at 4%. Compare this answer with the tabular value.

8. Expand the following expressions by using the binomial formula:

(*a*) $(a + b)^7$,
(*b*) $(1 + i)^4$,
(*c*) $(1 + 0.02)^5$,
(*d*) $(a + 1)^6$,
(*e*) $(x - 2y)^7$,
(*f*) $(\frac{1}{3}P + R)^4$,
(*g*) $(a^2 - b^2)^8$,
(*h*) $(5y - \frac{1}{2}x)^5$.

9. Using the binomial formula, compute correctly to two decimal places the amount of \$300 compounded annually, interest at 4%, at the end of 10 years. [*Hint:* $S = P(1 + r)^n$.]

10. Using the binomial formula, find the approximate numerical value of each of the following:

(a) $(101)^{\frac{1}{2}}$, (c) $\sqrt[3]{130}$, (e) $\sqrt[4]{83}$, (g) $\sqrt[3]{9}$,
(b) $(1.02)^5$, (d) $\sqrt{83}$, (f) $(1.01)^7$, (h) $\sqrt[5]{33}$.

11. When money is placed at compound interest, the rate actually earned is called the *effective rate* and is denoted by i. The stated rate per year is called the *nominal rate* and is denoted by j. When these two rates are equivalent, the two compound amounts must be equal. Write the equation expressing this fact for \$1 for m conversion periods per year.

12. Using the binomial formula, find the value of each of the following:

(a) $(101)^3$, (c) $(98)^3$, (e) $(51)^3$, (g) $(97)^2$,
(b) $(11)^4$, (d) $(0.011)^3$, (f) $(1{,}001)^2$, (h) $(1.01)^3$.

13. Find the next to the last term in the binomial expansion of each of the following:

(a) $(a + b)^{11}$, (c) $(2 + r)^{17}$, (e) $(3x - \frac{1}{2}y)^{10}$,
(b) $(y - x)^{13}$, (d) $\left(1 + \dfrac{j}{m}\right)^{12}$, (f) $(\sqrt{i} + j)^{15}$.

14. The coefficients of the binomial expansion of $(a + b)^n$ are, in order,

For $n = 0$ 1
For $n = 1$ 1 1
For $n = 2$ 1 2 1
For $n = 3$ 1 3 3 1

Extend this configuration up to $n = 6$ and thus obtain *Pascal's triangle* for n up to 6. What is the law of formation of any number in the triangle other than the units?

15. Since

$$\begin{aligned}(10y + 5)^2 &= 100y^2 + 100y + 25\\ &= 100y(y + 1) + 25,\end{aligned}$$

we can square any number ending in 5, mentally, by multiplying the figure in the tens' place by itself increased by 1 and then placing 25 following this product. Thus 35 is squared mentally as follows: $3 \times 4 = 12$, and followed by 25 gives 1,225.

Square mentally: 45, 85, 25, 55, 115, 105.

16. Since $(50 + y)^2 = 2{,}500 + 100y + y^2 = (25 + y) \times 100 + y^2$, we can square a number larger than 50 by finding its excess over 50, adding this to 25 to get the hundreds, and then adding the square of the excess. Thus

$$52^2 = (25 + 2)100 + 4 = 2{,}704.$$

Square mentally: 51, 53, 54, 55, 56.

17. Obtain a method in Problem 16 for numbers less than 50.

93. The Number *e*. A very important quantity in mathematics to which we shall give some attention here is one designated by the letter *e*, in honor of the great Swiss mathematician Euler (1707–1783). This number enters into the development of a formula used in compound interest problems when the interest is assumed to be compounded continuously and is important in simplifying computations in the business of large financial institutions. Another important use of this mathematical constant is in the field of statistics. At this point we shall confine our attention to finding its value.

The number *e* has as its value the limit of $(1 + 1/n)^n$ as n becomes infinite. By the binomial formula, using $a = 1$ and $b = 1/n$, we may write the expansion of $(1 + 1/n)^n$ as follows:

$$\left(1 + \frac{1}{n}\right)^n = 1 + n \times \frac{1}{n} + \frac{n(n-1)}{n^2 \times 1 \times 2} + \frac{n(n-1)(n-2)}{n^3 \times 1 \times 2 \times 3} + \cdots .$$

As n becomes large, the numbers $(n - 1)$, $(n - 2)$, and so forth, will not differ appreciably from n, so that the factors containing n's will tend to cancel out of each numerator and the corresponding denominator. In the limit, therefore, when n becomes infinite, the series simplifies, and we have according to our definition of the number *e*

$$\begin{aligned} e &= 1 + 1 + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \cdots \\ &= 1 + 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \cdots \\ &= 1 + 1 + 0.50 + 0.167 + 0.042 + \cdots . \end{aligned}$$

By taking enough terms in the series it can easily be found that the value of *e* correct to five decimal places is 2.71828.

PROBLEMS

1. Find the value of e rounded off to six decimal places.
Answer: 2.718282.

2. We have seen that the relationship between effective rate and nominal rate is expressed by the formula $1 + i = (1 + j/m)^m$. Let $j/m = k$, and show that $i = e^j - 1$ when the number of conversion periods, m, increases indefinitely. This formula gives the effective rate equivalent to a nominal rate, j, compounded continuously.

Exercises

1. Find L and S for the following sequences.

(*a*) 3, 11, 19, . . . to 10 terms,
(*b*) 5, 12, 19, . . . to 10 terms,
(*c*) $-4, -7, -10$, . . . to 14 terms,
(*d*) $12.50, $19.75, $27.00 . . . to 6 terms.

2. If a person invests $500 at the end of each year for 20 years at 2% simple interest, find the accumulated value of his investments at the end of 20 years.

3. Find the accumulated interest at the end of 10 years if a man invests $300 at the end of each year for 10 years at 3% simple interest.

4. Find the sum of each arithmetic series: (*a*) $825 + $850 + $875 + · · · + $1,000, (*b*) $1.12 + $1.24 + $1.36 + · · · + $2.08.

5. A debtor agrees to pay a $1,000 debt in the following manner: $100 and the simple interest at 4% on all unpaid principal at the end of each year for 10 years. Find the total sum of money paid by the debtor in discharging his debt.

6. What is the common ratio of each of the following geometric progressions?

(*a*) 3, 6, 12, 24, . . . ,
(*b*) 10, 5, 2.5, . . . ,
(*c*) 48, -12, 3, . . . ,
(*d*) $r, r^2, r^3, \ldots,$
(*e*) $r, r^3, r^5, \ldots,$
(*f*) $(1.03), (1.03)^2, (1.03)^3, \ldots,$
(*g*) $1, (1.04)^2, (1.04)^4, \ldots,$
(*h*) $(1.02)^{-7}, (1.02)^{-6}, (1.02)^{-5}, \ldots,$
(*i*) $(1 + i)^{-9}, (1 + i)^{-7}, (1 + i)^{-5}, \ldots.$

7. Find the sum of each of the following expressions:

(*a*) $1 + (1.04) + (1.04)^2 + (1.04)^3 + \cdots + (1.04)^8,$
(*b*) $1 + (1.02) + (1.02)^2 + (1.02)^3 + \cdots + (.102)^{10},$
(*c*) $1 + (1.01) + (1.01)^2 + (1.01)^3 + \cdots + (1.01)^{15},$
(*d*) $1 + (1.05) + (1.05)^2 + (1.05)^3 + \cdots + (1.05)^{27}.$

8. Expand the following expressions by using the binomial formula:

(*a*) $(1 + 0.03)^2$, (*d*) $(2a + b)^4$, (*g*) $(1 + z)^{10}$,

(*b*) $(1 + 0.05)^5$, (*e*) $(1.04)^6$, (*h*) $\left[\left(\frac{2x}{y} + z^2\right)\right]^3$,

(*c*) $(x + y)^5$, (*f*) $(1 + 1/1.06)^3$, (*i*) $(0.1p^{\frac{1}{3}} - 2)^4$.

9. Use the binomial formula to find the approximate numerical value of (*a*) $51^{\frac{1}{3}}$, (*b*) $(1.01)^8$, (*c*) $\sqrt{145}$, (*d*) $\sqrt[3]{28}$.

10. Find the amount and the compound interest due at the end of the time:

(*a*) \$1,000, 18 years at 2% compounded annually,
(*b*) \$1,500 22 years at $1\frac{1}{2}$% compounded annually,
(*c*) \$3,000, 12 years at 2% compounded quarterly,
(*d*) \$4,000, $6\frac{1}{2}$ years at 1% compounded semiannually,
(*e*) \$3,500, 8 years at 6% compounded monthly,
(*f*) \$7,000, 4 years 3 months at 2% compounded quarterly,
(*g*) \$2,600, 5 years 6 months at $2\frac{1}{2}$% compounded semiannually.

11. Find the present value on Jan. 1 of the following:

(*a*) \$1,000 due on Jan. 1, 10 years hence; 2% compounded semiannually,
(*b*) \$2,000 due on July 1, $15\frac{1}{2}$ years hence; 2% compounded quarterly,
(*c*) \$500 due on Apr. 1, $6\frac{1}{4}$ years hence; 2% compounded annually,
(*d*) \$1,200 due on Oct. 1, $8\frac{3}{4}$ years hence; 2% compounded monthly.

12. Discount \$600 for $8\frac{1}{2}$ years at 5% compounded monthly.

13. What nominal rate, converted quarterly, will be equivalent to an effective rate of 6%, converted annually? [*Hint:* $1 + i = \left(1 + \frac{j}{m}\right)^m$, in which i = effective rate, j = nominal rate, and m = number of conversion periods.]

14. What effective rate will be equivalent to a nominal rate of 2%, converted quarterly? [*Hint:* See Problem 13.]

15. Compare the simple interest on \$300 for 6 months at 2% with the compound interest on the same sum for 6 months at 2%, compounded quarterly.

16. For a certain year the population of a certain city was 123,000. Two years later, the population had increased to 132,500. What was the average monthly rate of increase for the 2-year period?

17. Solve the equation $i = e^j - 1$ for j when $i = 0.06$. [*Hint:* $0.06 = e^j - 1$; hence; $e^j = 1.06$. Take logarithms of both sides, and use 2.718 as the value of e.]

18. A building cost \$60,000. At the end of each year, the owners deduct 10% of its value as estimated at the beginning of the year. What is the estimated value at the end of 8 years?

19. A machine has been depreciated 6% per year for the past 5 years. Its present value is \$350. What was the cost of the machine?

20. Jack Green borrows \$4,000 with interest at 6%, compounded monthly. He agrees to discharge the loan by equal monthly payments on an amortization plan. Find the amount of each payment if the loan is to be paid in 8 years. Construct an amortization schedule for the first 5 payments.

21. A bond issue of \$200,000 is to be retired in 20 equal annual installments and bears simple interest at 4%. How much interest will be paid on the issue during its life?

22. An automobile costing \$3,000 is found to be worth only \$1,500 on the second-hand market 3 years later. Find the fixed annual percentage of diminishing value.

23. A salesclerk started working for a chain store for \$2,400 the first year, and was given a \$100 yearly increase for 15 years. What was his salary 11 years after he started working? How much had he been paid in the 11 years?

24. How many numbers between 50 and 500 are exactly divisible by 13?

25. An office building consists of 30 stories, each 12 ft in height. How many miles would an elevator operator travel in taking a person to each floor in turn, returning to the ground floor after each floor stop?

26. Assuming that, under certain conditions, the bacterium causing milk to sour divides each half hour, determine the number of bacteria originally present in milk which has a bacteria count of 500,000 per cubic centimeter at the end of a 5-hr period.

27. A college track star trains for the mile run by running 440 yd the first day and 30 yd further each day thereafter until on the last day he runs the mile. How far does he run altogether?

28. A certain loan company loans \$200 to be paid back in 18 monthly installments of \$12.91. Would it be cheaper for the borrower to repay it in 18 equal monthly installments of \$11.11 plus simple interest at 12% on the unpaid balance? How great is the difference?

29. New York City had approximately 2,000,000 inhabitants in 1880 and 7,500,000 in 1940. Find the average percentage increase per decade. Estimate the 1950 population on the basis of this average increase.

30. A corporation sets aside \$40,000 a year to retire a bond issue due in 12 years. If money is worth 1% converted annually, find the amount of the bond issue.

31. A firm wishes to raise \$500,000 through issuing 2% sinking fund bonds. If it can set aside \$75,000 a year toward the retirement of the bonds and for the annual interest payments (2% of \$500,000), when should the bonds mature if money is worth 1%?

32. Use the formula for amount of an annuity, and solve for the indicated letters:

	R	r	n	S
(*a*)	500	0.02	12	———
(*b*)	200	0.03	15	———
(*c*)	1,000	0.04	—	10,000
(*d*)	———	0.01	20	25,000

Self-Test

Time: 50 Minutes

(Score 5 points for each problem correctly solved)

In an arithmetic progression, given $a = 4$, $d = 3$, $n = 6$. (Data for Problems 1, 2, 3.)

1. Find the last term.

2. Find the sum of all the terms.

3. What is the value of the $(n - 1)$th term?

In a geometric series, given $a = 10$, $r = \frac{2}{5}$. (Data for Problems 4, 5.)

4. Find the 5th term.

5. Find the sum of 5 terms.

Given the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. (Data for Problems 6, 7, 8.)

6. Find the sum of 8 terms.

7. Write the expression for the general term.

8. Find the limit of the sum of the series as the number of terms becomes infinite.

9. Use the tables and find the value of the sum of the first 5 terms of the series $(1.06) + (1.06)^2 + (1.06)^3 + \cdots$.

10. Write the 5th term of the binomial expansion for $(p + q)^9$.

11, 12. Use 3 terms of the binomial formula, and compute an approximation to the cube root of 126.

13, 14. Find the compound amount of $280 at 2% for 3 years 6 months, compounded quarterly.

15, 16. In how many years will $2,000 amount to $2,901.89 if compounded annually at $1\frac{1}{2}$%?

17, 18. What simple interest rate is equivalent to a compound interest rate of 1% compounded semiannually?

19, 20. A man purchases a house for $10,000. How much of a payment should he make each year, starting 1 year hence, in order to retire his debt in 20 payments if the interest rate is 3%?

Chapter IX

EQUATIONS AND BUSINESS APPLICATIONS

Introduction. In Chapter II, we introduced simple equations of the first degree and discussed the steps involved in solving them. In the present chapter, we shall review our earlier work while employing the conventional symbols of several business fields, including banking, insurance, and related fields. This will be followed by sections devoted to simultaneous equations, quadratic equations, exponential equations, and equations of value. Since modern business involves a great deal of mathematical work, a thorough understanding of the use of equations is indispensable.

94. First-degree Equations. The steps involved in solving simple equations of the first degree bring into play such operations as clearing of fractions, collecting terms, and transposing. The algebraic processes remain the same no matter how complicated the nature of the symbolism may be. The following illustrations and problems will serve to review our earlier work and to give practice in solving certain equations frequently encountered. It will be observed that in each of these equations the unknown for which the equation is solved is raised to the first power. Such equations are examples of *first-degree* equations and are said to be *linear* in the unknown.

Illustrations:

1. Solve for R in $R(1 + i)^n - Si - R = 0$.

Transpose so that only terms involving R are on the left,

$$R(1 + i)^n - R = Si.$$

Factor R out of both terms on the left,

$$R[(1 + i)^n - 1] = Si.$$

Divide both sides by $[(1 + i)^n - 1]$, the coefficient of R,

$$R = \frac{Si}{[(1 + i)^n - 1]}.$$

We observe that the manner of solving this equation for R is in no way different from that of a simpler form, say, $aR - Q - R = 0$ or, perhaps, $2R - 8 - R = 0$.

2. Solve for C in $A_n = Rv^n + rCa_{\overline{n}|i}$.

Transpose so that the term involving C is on one side and the others are on the other side,

$$-rCa_{\overline{n}|i} = Rv^n - A_n.$$

Change signs throughout,

$$rCa_{\overline{n}|i} = A_n - Rv^n.$$

Divide both sides by $ra_{\overline{n}|i}$, the coefficient of C,

$$C = \frac{A_n - Rv^n}{ra_{\overline{n}|i}}.$$

Compare the solution of the equation above with that of the following: Solve for x in

$$14 = 8 + 2x.$$

Notice that here, as above, we may obtain the solution by transposing, changing signs, and dividing both sides by the proper coefficient. It should be noted in Illustration 2 that $a_{\overline{n}|i}$ (read "a angle n at i") is one symbol, indivisible, and the n and i cannot be separated from the a.

PROBLEMS

1. Solve the equation of Illustration 1 for S.
2. Solve the equation of Illustration 2 for $a_{\overline{n}|i}$.
3. Solve for S in $D = \dfrac{C - S}{n}$.
4. Solve for i in $T = D + iD_k + iB_k$.
5. If ${}_nC_r$ is a symbol denoting the number of combinations of n things taken r at a time and ${}_nP_r$ stands for the number of permutations of n things taken r at a time, how many symbols are involved in the formula $r!\,{}_nC_r = {}_nP_r$?

6. Solve the formula of Problem 5 for ${}_nC_r$. (*Note:* The n and r cannot, under any circumstances, be separated from the C or the P.)

7. Solve the equation ${}_nE_x = v^n p_x$ for p_x.

8. Solve for x in the equation $x - 4(6 + x) = 18 + 7x$.

9. Replace the symbol x in Problem 8 by the symbol M_x, and solve the equation for M_x. Are the steps involved the same as they were when the single-letter symbol was used?

10. Solve for v in the equation $C_{x+1} = vD_{x+1} - D_{x+2}$.

11. Solve for M_x in the equation $P_x = \dfrac{M_x}{N_x}$.

12. The net annual premium of an n-year term insurance policy is denoted by the symbol ${}_nP_x$. Solve the following equation for the net annual premium.

$${}_nP_xN_x - M_x = N_{x+n}\,{}_nP_x - M_{x+n}.$$

13. Solve the following formula for i:

$$a_{\overline{n}|i} = \frac{1 - (1.03)^{-n}}{i}.$$

14. Solve the following formula for ${}_rV_x$:

$$P_x = \frac{A_{x+r} - {}_rV_x}{a_{x+r}}.$$

15. Solve the following equation for x:

$$(M_1 + M_2 + M_3 + \cdots + M_r)(1 - xi) = M_1(1 - n_1 i) + M_2(1 - n_2 i) + M_3(1 - n_3 i) + \cdots + M_r(1 - n_r i).$$

16. Solve for R in

$$S = R\,\frac{(1 + i)^n - 1}{i}.$$

17. Solve for h in the equation $p = \dfrac{h}{h + f}$.

18. Solve for h in the equation $q = \dfrac{f}{h + f}$.

19. Solve $r = \dfrac{\Sigma d_1 d_2}{n\sigma_1\sigma_2}$ for n. (*Note:* $\Sigma d_1 d_2$ is one symbol and should not be separated.)

20. Solve $y^2 - 3y + 2x = 4$ for x.

21. Solve $tx - \frac{1}{2}x + 3 = 2tx$ for x; for t.

22. Solve $3!\,2!\,P = 6!$ for P.

23. Solve $A = \frac{1}{2}(b_1 + b_2)h$ for b_1.

24. Solve $V = (\frac{1}{3})h(b_1 + b_2 + b_1b_2)$ for b_1.

25. Solve $Av^3 = A_1v + A_2v^2 + A_3v^3$ for A.

26. Solve $A = 2rh + \frac{1}{2}\pi r^2$ for h.

27. Solve $\dfrac{w(u - v)}{3} + w = \dfrac{2u}{3} - v$ for w.

Solve each of the following for x:

28. $x(3a - 2) = (\frac{3}{2})a - 1.$

29. $\dfrac{3(x + 4)}{13} = 2x - 1.$

30. $2[3x - (1 + x)] = 5x + 8.$

31. $x - 1 = \dfrac{2x}{5} + x - 5 + 3(x + 1).$

32. $\dfrac{x}{3}[9 + (2x - 1) + x - (3x - 7)] = 3x - 1.$

33. $\dfrac{x - 1}{x} = \dfrac{6}{7}.$

34. $\dfrac{x - 1}{x + 1} = -1.$

35. A man invests \$5,500 at 2%. How much additional money should he invest, some at 1% and $1\frac{1}{2}$ times as much at $2\frac{1}{2}$%, to realize a total yearly interest of \$205?

36. Four thousand dollars of the net profit of a company is paid out as employees' bonus; 13% of the net profit, after deducting the bonus, is deducted for tax. What is the net profit of the company before the tax and bonus deductions if the bonus is to be 15% of the net profit after the tax deduction?

37. In a dairy, the ice cream contains 15% cream. If the mixture is to be made from coffee cream of 20% butterfat and milk which tests 5%, what portion of each will have to be used in a mixture of 100 lb?

38. How many pounds of seed worth 42¢ per pound must be mixed with other seed worth 50¢ per pound to make a 50-lb mixture worth 45¢ per pound?

39. In making some candy a "homemade" sweetshop uses a mixture of 70% sugar at 6¢ a pound and 30% corn sirup at 3¢ per pound. If sugar advances to 8¢ per pound, what must be the ratio of sugar and corn sirup used if the cost of production is to remain the same?

40. A man has $10,000 invested at 3%. How much must he invest at 4% to make his total income equal to $3\frac{1}{4}$% on his total investment?

41. The cold-water tap can fill a bathtub in 4 min. The hot-water tap can fill it in 6 min. The drain can empty the tub in 8 min. If both taps are on and the drain is open, how long will it take to fill the tub?

95. Simultaneous Equations. A man has a total of $500 invested in two stocks, one paying 5% and the other 6% annually. If the 6% investment yields annually $3 less than the 5% investment, what is the amount in each stock?

In this problem there are two unknown quantities, namely, the two amounts invested; and we seek first to formulate two equations that may express the conditions of the problem algebraically. Then, by solving these equations for the values of the unknowns, we may find the *solution* of our problem. We may proceed as follows: Let

x = amount invested in the 5% stock,

and y = amount invested in the 6% stock.

Then since the sum of the two investments is $500, we may write the equation

$$x + y = \$500.$$

The 5% stock yields $0.05x$, and the 6% stock yields $0.06y$; and the difference in these yields is $3. Therefore, we may write a second equation

$$0.05x - 0.06y = \$3.$$

Each of the two equations that arise out of the conditions of the problem is a first-degree equation in the unknowns x and y. A solution to our problem will consist of a pair of values that will satisfy both equations. In arriving at our solution, we say that we solve the equations as *simultaneous equations*. The solution may be obtained in several ways, but we shall confine our discussion to a rather simple method known as the method of *elimination*. The steps in the solution of the problem at hand are as follows:

Bring together our two equations, and write

$$\begin{cases} x + y = 500, & (1) \\ 0.05x - 0.06y = 3, & (2) \end{cases}$$

in which the brace is used to indicate that the two equations are to be treated as simultaneous equations.

Multiplying the members of (1) by 0.06 and pairing the resulting equation with equation (2), we obtain

$$0.06x + 0.06y = 30, \qquad (3)$$

$$0.05x - 0.06y = 3. \qquad (2)$$

Adding (2) and (3), we obtain

$$0.11x = 33,$$

from which

$$x = 300. \qquad (4)$$

Upon multiplying (1) by 0.05 and pairing the resulting equation with (2), we obtain

$$0.05x + 0.05y = 25, \qquad (5)$$

$$0.05x - 0.06y = 3. \qquad (2)$$

Subtracting (2) from (5), we obtain

$$0.11y = 22,$$

from which

$$y = 200. \qquad (6)$$

Upon substituting $x = 300$ and $y = 200$ into equations (1) and (2), we obtain

$$300 + 200 = 500,$$

$$0.05(300) - 0.06(200) = 15 - 12 = 3.$$

Hence, we have found that \$300 invested in 5% stock and \$200 invested in 6% stock is a solution to our problem.

We must remember that in the process of solving simultaneous equations whatever is done to one side of one equation must also be done to the other side. It is because of this principle that we are able to multiply both members of an equation by suitable numbers and thus be in a position to eliminate one of the unknowns by addition or subtraction. It should be pointed out, too, that all the rules pertaining to clearing of parentheses, transposing, collecting terms, dividing by the coefficient of the unknown, and so forth, that are used in solving simple equations of the first degree in one unknown apply equally well in the process of solving simultaneous equations of the first degree.

Illustration: Solve for x and y,

$$\begin{cases} 5x - 3y = 9, & (1) \\ 3x + y = 11. & (2) \end{cases}$$

Multiply equation (2) by 3, and add,

$$\begin{aligned} 5x - 3y &= 9 && (1) \\ \underline{9x + 3y} &\underline{= 33} \\ 14x &= 42 \\ x &= 3. \end{aligned}$$

Substitute 3 for x in equation (2),

$$\begin{aligned} 3(3) + y &= 11, \\ 9 + y &= 11, \\ y &= 11 - 9, \\ y &= 2. \end{aligned}$$

Hence, the indicated solution of the pair of simultaneous equations is $x = 3$, $y = 2$.

We may check that these values actually furnish the solution by substituting them into the two given equations.

$$\begin{aligned} 5(3) - 3(2) &= 9, & 3(3) + 2 &= 11, \\ 15 - 6 &= 9, & 9 + 2 &= 11, \\ 9 &= 9; & 11 &= 11. \end{aligned}$$

There is no fixed rule observed in the manner in which we eliminate an unknown. For example, we could have eliminated x first.

$$\begin{array}{lrcl} \text{Multiply equation (1) by 3,} & 15x - 9y &=& 27 \\ \text{Multiply equation (2) by 5,} & 15x + 5y &=& 55 \\ \hline \text{Subtract,} & -14y &=& -28 \\ & y &=& 2. \end{array}$$

PROBLEMS

1. Plot equations (1) and (2) of the preceding illustration on the same graph paper (see Section 74). Interpret your results.

2. Solve for x and y, and check your results.

$$\begin{cases} x - 2y = -6, \\ 4x + 9y = 61. \end{cases}$$

Answer: $x = 4$, $y = 5$.

3. A thousand dollars is invested in three different securities. One pays 3% interest, one 4% interest, and the other 5% interest. The total income from the investments is \$43, and the income on the 4% securities is twice that on the 3% securities. How much is invested in each security? [*Hint:* Let x = part paying 3%, y = part paying 4%. Then, $1{,}000 - (x + y)$ = part paying 5%. Set up two equations according to the conditions of the problem, and solve for x and y.]

Answer: 200 at 3%; 300 at 4%; 500 at 5%.

When there are three equations and three unknowns, we may eliminate one unknown between any two of the equations and thus obtain one equation in two unknowns. Then we may eliminate the same unknown between any other pair of the equations and obtain another equation in two unknowns which is paired with the one just obtained. The two equations may then be solved for their two unknowns in the manner shown in the preceding illustration, and the third unknown is found by substituting these two values into any one of the three original equations.

PROBLEMS

Solve the following simultaneous equations, and check the results:

1. $\begin{cases} x + 3y = 14, \\ 2x + y = 8. \end{cases}$

2. $\begin{cases} 2A - B = 1.09, \\ 3A + 6B = 9.18. \end{cases}$

3. $\begin{cases} P + 200r = 112, \\ 2P + 500r = 130. \end{cases}$

4. $\begin{cases} C + 0.30P = \$96, \\ C - 0.20P = \$86. \end{cases}$

5. $\begin{cases} 3x + 2y - z = 4, \\ x + y + 3z = 12, \\ 5x - 3y + 2z = 5. \end{cases}$

6. $\begin{cases} \dfrac{2}{3 - y} = \dfrac{6}{8x - 5}, \\ \dfrac{4}{5 - y} = \dfrac{4}{3 - x}. \end{cases}$

7. $\begin{cases} x + y + z = 12, \\ 2x - y + z = 8, \\ 3x + 2y - z = 3. \end{cases}$

8. $\begin{cases} x + y - z = 3, \\ x - y + z = 2, \\ 2x + 2y - z = 6. \end{cases}$

9. $\begin{cases} 0.6x - 0.3y = 2.7, \\ x + 0.5y = 3.5. \end{cases}$

10. $\begin{cases} 1.3y - 0.7x = 3, \\ 2.1y + 0.4x = 12.5 \end{cases}$

11. $\begin{cases} 2x - z = 4, \\ 3y + 2z = 20, \\ 4x - 3y = 4. \end{cases}$

12. $\begin{cases} 3x + 4y - 5z = -4, \\ 5x + z = 8, \\ 2x - 7y = -12. \end{cases}$

13. $\begin{cases} 0.2x - 1.4y + z = 4.6, \\ 0.7x + 2.3y - 2z = -3.8, \\ 0.3x - y + 1.4z = 4.6. \end{cases}$

14. $\begin{cases} 3x - 40y + 7z = 6, \\ 2x + 20y - 3z = 20, \\ -5x + 17y + 2z = -5. \end{cases}$

15. $\begin{cases} 2y - x + 4z = 19, \\ 5x - 4y + 6z = 15, \\ 2x + 13y - 9z = -25. \end{cases}$

16. $\begin{cases} 13x + 21y = 9, \\ 4x - 8y = 10. \end{cases}$

17. $\begin{cases} 3w - 4z = 18, \\ 7w + 3z = 5. \end{cases}$

18. $\begin{cases} \frac{5}{3}x + \frac{1}{4}y = \frac{13}{8}, \\ \frac{1}{2}x - \frac{2}{3}y = -\frac{5}{8}. \end{cases}$

19. $\begin{cases} \frac{5}{6}x - \frac{1}{5}y = -\frac{2}{3}, \\ \frac{3}{2}x + \frac{2}{3}y = -\frac{19}{3}. \end{cases}$

20. Show that the pair of equations $3x - y = 7$ and $6x - 2y = 3$ cannot be solved. Why is there no unique solution whenever the coefficients of the x terms are in the same ratio as the coefficients of the y terms?

21. Solve the pair of equations $\dfrac{1}{x} + \dfrac{2}{y} = 2$ and $\dfrac{2}{x} - \dfrac{3}{y} = \dfrac{1}{2}$ by using $\dfrac{1}{x}$ and $\dfrac{1}{y}$ as the variables.

22. Solve: $\begin{cases} \dfrac{3}{x} - \dfrac{2}{y} = \dfrac{1}{3}, \\ \dfrac{4}{x} + \dfrac{3}{y} = \dfrac{4}{3}. \end{cases}$

23. Solve: $\begin{cases} \dfrac{5}{x} + \dfrac{3}{y} = -\dfrac{3}{2}, \\ \dfrac{11}{x} + \dfrac{4}{y} = \dfrac{25}{6}. \end{cases}$

24. An income of \$48 is realized on two loans which total \$1,350. One of the loans pays 3% interest, and the other pays 4%. How much is in each loan?

25. A broker was forced to sell 100 shares of stock at \$63 and thereby took a loss of 25%. If he had paid \$80 for some of the shares and \$90 for the others, how many shares were bought at each price?

26. A 1% investment yields annually \$84 less than a $2\frac{1}{2}$% investment. The total of the two investments is \$8,260. How much is the amount of each?

27. The assets of the Johnson Manufacturing Company exceeded the liabilities by \$6,000. If the liabilities had been twice as large, the excess of the assets would have been only \$4,000. What were the company's assets and liabilities?

28. The daily pay roll of a factory is \$2,543. If 249 men are employed, some at \$11 and the rest at \$9 a day, how many men are employed at each rate?

29. Clark invests \$30,000, some at 3% and the remainder at $3\frac{1}{2}$%. His first year's interest is \$944.50. How much did he invest at each rate?

30. A sum of money invested at simple interest for 9 months amounts to \$121.22, and a like sum invested for 6 months at a rate $\frac{3}{4}$% less amounts to \$119.04. Find the number of dollars in each investment and the rates of interest.

31. A department manager of a corporation gets 10% of the profits, from which the Federal income tax and his commission salary are first deducted as expenses. The profits are \$20,000 before deductions are made. Suppose the income tax rate is 12% and an exemption of \$3,000 is allowed for net income below \$25,000. Find the salary and tax. (*Note:* The salary can be deducted as an exemption in paying the tax.)

32. *A* and *B* were partners in a small firm and agreed to share the profits in proportion to the capital invested. The profits were \$4,000. *A* owned a third interest plus \$200, and his share of the profits was \$1,400. What was the value of the business and the capital of each partner?

33. The manager of a paint shop is to receive a bonus of 5% of the net income after the Federal income tax of 25% and his bonus

are deducted. The profit without deductions is $22,000. Find the bonus and the tax. (*Note:* The bonus is deductible from the net income for taxation purposes.)

34. It is shown by the consolidated balance sheets of A and B companies that Company A owns 50% of the capital stock of Company B and that Company B owns 25% of the capital stock of Company A. The balance sheets of the companies at the end of the fiscal year give the net assets of Company A as $1,200,000 and of Company B as $650,000. Find the true net worths of the companies.

35. Mr. Jackson borrows a sum of money for a period of time at 6% simple interest and pays back $103. He also borrows the same sum for a 3-month-longer period at $5\frac{1}{2}$% simple interest and pays back $104.12. Find the sums borrowed and the periods of time.

36. Admission to a play at the Little Hamlet Summer Theater was as follows: children, 60¢; adults, $1. The receipts for a sellout of 600 tickets were $500. How many tickets of each kind were sold?

37. Si and Alf each own some sheep. If Si gives a sheep to Alf, then Alf has twice as many as Si; but if Alf gives a sheep to Si, then they both have the same number. How many sheep does each man own?

38. In 13 hr, B walks 7 miles more than A does in 9 hr; in 8 hr, A walks 12 miles more than B does in 7 hr. Find their rates of walking.

39. A, B, and C each have savings which together amount to $3,400. If B gives A $500, their two savings are just reversed. B has as much more than C as is equal to $\frac{6}{10}$ of A's savings. How much savings does each man have?

96. Quadratic Equations. The owner of a fruit store constructs a rectangular window which admits light through 32 sq ft of area. Afterward, he observes that the perimeter of the chromium frame is 24 ft. What are the dimensions of the window?

Let $x =$ length of one side.

Then $12 - x =$ length of the other side.

Hence, $x(12 - x) = 32,$

or $12x - x^2 = 32.$

Our task now is to find the value of x in this last equation. We observe that it is not of the first degree (linear) since it involves x to an exponent higher than 1. In fact, this equation is an example of an equation of the form $ax^2 + bx + c = 0$, which is called a *quadratic equation in x*.

In our problem, we have, after transposing terms,

$$x^2 - 12x + 32 = 0.$$

Factoring the left member, we may write

$$(x - 8)(x - 4) = 0,$$

Then applying the principle that a product equals zero when and only when *at least one of the factors is zero*, we may write

$$x - 8 = 0, \qquad \text{or} \qquad x = 8;$$

$$x - 4 = 0, \qquad \text{or} \qquad x = 4.$$

Check: $$12(8) - (8)^2 = 32,$$

and $$12(4) - (4)^2 = 32.$$

Hence, the dimensions of the window are 8 by 4 feet.

If factoring is not convenient, the quadratic equation $ax^2 + bx + c = 0$ may be solved by use of the well-known *quadratic formula*,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In our problem, $a = 1$, $b = -12$, $c = 32$. Upon substituting, we have

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(1)(32)}}{2(1)},$$

or $$x = \frac{12 \pm \sqrt{144 - 128}}{2} = \frac{12 \pm \sqrt{16}}{2} = \frac{12 \pm 4}{2}.$$

Hence, $x = 8$ and $x = 4$.

PROBLEMS

Solve and check each of the following equations:

1. $x^2 + 7x - 8 = 0$.
2. $x^2 - 8x - 5 = 0$.
3. $x^2 - 3x + 1 = 0$.
4. $2x^2 + 5x - 2 = 0$.
5. $(x + 1)^2 = 36$.

6. The interest on \$100 for 1 year compounded semiannually at a certain rate is \$2.52. Set up a quadratic equation, and find the rate.

7. A sheet of paper is to have a margin of 1 in. surrounding 24 sq in. of printing. What must be the dimensions of the sheet if it is to be twice as long as it is wide?

8. Draw a graph of the parabola $y = -x^2 - 12x + 20$. What interpretation can you give to the values of x where the curve crosses the x axis? Where does the curve appear to have a maximum? Can you discover some method by which one might go about finding precisely the maximum value of a quadratic function? What test would you apply for finding a minimum? [*Hint:* At a maximum or a minimum point there must be only one value of x where y has a definite value.]

9. A rectangular display area 8 by 10 ft is to be enlarged so as to give 45 sq ft more area by adding borders along two adjacent sides in such a way as to preserve the original shape. What must be the width of the narrower border?

10. Find the nominal interest rate j which for interest compounded semiannually is equivalent to a nominal interest rate of 4% compounded quarterly.

11. Find two equal successive trade discount rates that are equivalent to a single discount rate of 30%.

12. Find two equal successive trade discount rates that are equivalent to a single discount rate of 36%.

13. A rectangular lot has an area of 5,600 sq ft and is 100 ft longer than it is wide. What are the dimensions of the lot?

14. The radius of a circular flower exhibit is 5 ft. What increase in the radius will be required for an increase of 11π sq ft in the area?

97. Exponential Equations. In many problems we are confronted with the task of solving for the unknown when that unknown appears as an exponent. We encountered this type of problem, for example, in compound interest problems, where we were required to solve for the number of interest conversion periods, and in solving for the number of terms in a geometric series when certain other elements were given. An equation in which the unknown is involved in an exponent is called an *exponential equation.*

The manner of solving many equations of this type has already been pointed out in Chapters VII and VIII. We have seen that the basic thing to do is to take logarithms and thus use the exponent as a multiplier. After making this change in the form of the equation, we solve for the unknown in the usual manner.

Illustration 1: Solve $2^x = 25$ for x.

Take the logarithm of each side, $\log 2^x = \log 25$,
From the rule for the logarithm of a power, $x \log 2 = \log 25$,
From the logarithm table, $0.30103x = 1.39794$,
Divide by the coefficient of x, $x = 4.644$
(correct to three decimal places).

Problem

Solve $(1.06)^n = 1.79$ for n.
Answer: 10.0 (correct to one decimal place).

Illustration 2: Solve $5(3)^{n-1} = 1{,}215$ for n.
Take the logarithm of each side,

$$\log 5(3)^{n-1} = \log 1{,}215.$$

From the rule for the logarithm of a product,

$$\log 5 + \log 3^{n-1} = \log 1{,}215.$$

From the rule for the logarithm of a power,

$$\log 5 + (n - 1) \log 3 = \log 1{,}215.$$

From the logarithm table,

$$0.69897 + (n - 1)(0.47712) = 3.08458.$$

Transposing, $0.47712(n - 1) = 3.08458 - 0.69897.$

Collecting, $0.47712(n - 1) = 2.38561.$

Dividing by the coefficient of $(n - 1)$,

$$n - 1 = 5.000.$$

Transposing and collecting,

$$n = 6.000.$$

PROBLEM

Solve $4 = 64(0.5)^{2x}$ for x.

Answer: 2.

Illustration 3: In how many years will \$5,000 amount to \$8,000 if compounded semiannually at 8%?

$$S = P(1 + r)^n, \qquad 8{,}000 = 5{,}000(1.04)^n,$$

where n is the number of interest conversion periods. Take the logarithm of each side,

$$\begin{aligned} \log 8{,}000 &= \log 5{,}000 + n \log (1.04), \\ 3.90309 &= 3.69897 + 0.01703n, \\ 0.01703n &= 0.20412, \\ n &= 11.99, \text{ or } 12 \text{ (approximately).} \end{aligned}$$

Hence, there would be required approximately 12 periods of $\frac{1}{2}$ year each, or 6 years.

PROBLEM

In how many years will \$300 amount to \$500 at 2% interest compounded quarterly?

Answer: 25.56 years.

The exponential equation $y = Ke^{Cx}$, in which K and C are positive constants, is sometimes called the *snowball law, the law of organic growth,* or *the compound interest law.* This important relation is used in certain phases of finance and in many other fields where growth is being studied. The related curve $y = Ke^{-Cx}$, called *the law of organic decay,* is commonly used in the physical world in studying such phenomena as the decomposition of radium and the transmission of light. The exponential equation $y = Ke^{-Cx^2}$, known as the equation of the *normal curve,* or the *normal probability function,* is of such importance in the field of statistics that special attention to some of its properties and uses will be given in the next chapter, which deals with further statistical methods.

PROBLEMS

1. Plot the graph of each of the following equations (take $e = 2.718$): (*a*) $y = e^x$, (*b*) $y = e^{-x}$, (*c*) $y = 2e^x$, (*d*) $y = 2e^{-2x}$.

2. The amount of any principal P after n years with interest compounded *continuously* at the annual rate of $r\%$ is given by the formula $A = Pe^{nr}$. Draw a graph which shows the amount of \$100 compounded continuously at an annual rate of 2%. (Take $n = 0, 1, 2, \ldots, 20$.)

3. If a property whose present value is P is assumed to decrease in value continuously at a rate of $r\%$ per year for a period of n years, its value at the end of n years is given by the formula $V = Pe^{-nr}$. Draw a graph showing the value of a \$10,000 building assumed to depreciate at the rate of 8% per year over a 10-year period. What is the value of the building at the end of the period?

Exercises

Solve for x:

1. $2^{x-3} = 32$.

2. $2(1.04)^x = 3.364$.

3. $1.05^{-x} = 0.7945$.

4. $1.03^{-x} = 0.4120$.

5. $\left(1 + \dfrac{0.06}{2}\right)^x - 1 = 0.0609$.

6. $\dfrac{(1.06)^x - 1}{0.06} = 0.0609$.

7. Solve $L = ar^{n-1}$ for n.

8. Solve $s = \dfrac{ar^n - a}{r - 1}$ for n.

9. If $S = 0.98e^{0.53t}$, find t when $S = 11.72$. (*Note:* Take e as 2.718.)

10. The sum of \$80.25 was allowed to accumulate at $4\frac{1}{2}\%$ interest compounded annually until it had amounted to \$100. How many years was the money at interest?

11. An insurance policy for \$10,000 dated Jan. 1, 1955, is left in trust for a boy until it amounts to \$12,000. If the rate paid by the insurance company is $3\frac{1}{2}\%$, in what year can the boy expect to receive the \$12,000?

12. In how many years will a sum of money triple itself at 2% compounded semiannually?

13. Find the amount of \$1,000, interest at 3% per year for 10 years compounded continuously. Compare the result with the amount of \$1,000 at 3% for 10 years compounded annually.

98. Equations of Value. A merchant owes a manufacturer three debts: \$800 due in 1 year, \$600 due in 2 years, and \$400 due in 3 years. He desires to pay all three debts in a single payment of \$1,800 at some future date. If money is worth 2%, when can the single payment of \$1,800 be made so as to make an equitable settlement?

Our problem here is to find a time, short of the expiration of the 3 years, at which the payment of \$1,800 would be equivalent, for both the debtor and the creditor, to the arrangement of the three unequal payments due at different dates. If the total debt were paid immediately, the arrangement would be unfair to the merchant, for he would be denied the use of the money over the extended period of time. On the other hand, a settlement at the end of the 3 years would mean that the manufacturer would be without a part of the money due him for too long a period of time.

We shall let n denote the time in years to the date at which the single payment of \$1,800 is to be made. The principal necessary to accumulate to \$1,800 in n years can be expressed by solving the compound interest formula for P. We have

$$S = P(1 + r)^n,$$

or

$$P = \frac{S}{(1 + r)^n}.$$

Substituting for S and r, we have $P = \dfrac{\$1,800}{(1.02)^n}$.

Since P represents the sum of money that would yield \$1,800 when compounded at 2% for n years, it is the *present value* of \$1,800 at 2% for n years.

The \$800 debt is due in 1 year, the \$600 debt in 2 years, and the \$400 debt in 3 years. Consequently, the *present value* of each debt is $\dfrac{\$800}{(1.02)^1}$, $\dfrac{\$600}{(1.02)^2}$, and $\dfrac{\$400}{(1.02)^3}$, respectively.

In order to be assured that the single payment of \$1,800 is entirely equivalent to the payment of the three separate debts at the times specified, we must have the *present value of the single sum equal to the present values of the three debts.* That is, we must have

$$\frac{\$1,800}{(1.02)^n} = \frac{\$800}{1.02} + \frac{\$600}{(1.02)^2} + \frac{\$400}{(1.02)^3}. \tag{A}$$

Such an equation is known as an *equation of value.* The value of n which satisfies this equation will be the time ("equated time") at which the single payment is equivalent to the payment of the three debts separately. We proceed with the solution as follows:

$$\begin{aligned}\frac{\$1,800}{(1.02)^n} &= \$800\,\frac{1}{1.02} + \$600\,\frac{1}{(1.02)^2} + \$400\,\frac{1}{(1.02)^3}\\ &= \$800\,\frac{1}{1.02} + \$600\,\frac{1}{1.0404} + \$400\,\frac{1}{1.061208}\end{aligned}$$

$$= \$800(0.980392) + \$600(0.961169) + \$400(0.942322)$$

$$= \$784.3136 + \$576.7014 + \$376.9288.$$

$$\frac{\$1800}{(1.02)^n} = \$1,737.94.$$

Cross multiply, $1,800 = 1,737.94(1.02)^n$.

Take the logarithm of each side,

$$\log 1,800 = \log [1737.94\ (1.02)^n].$$

From the rules of logarithms,

$$\log 1,800 = \log 1737.94 + n \log (1.02).$$

From the tables,

$$3.25527 = 3.24002 + 0.00860n.$$

Transposing and collecting,

$$-0.00860n = -0.01525.$$

Divide by the coefficient of n,

$$n = 1.77.$$

The date on which the single payment can equitably settle the three debts is 1.77 years, or approximately 1 year 9 months in the future.

Let us now introduce symbols to represent the different quantities which entered into the *equation of value* of the example and then write a general equation which may be used for solving problems of the type above, frequently encountered in commuting financial obligations.

Let M_1, M_2, M_3 stand for the debts \$800, \$600, and \$400, respectively. Also, let

$$\frac{1}{1.02} = v,$$

so that

$$\frac{1}{(1.02)^2} = v^2,$$

$$\frac{1}{(1.02)^3} = v^3,$$

and

$$\frac{1}{(1.02)^n} = v^n.$$

Then, since $S = M_1 + M_2 + M_3$, we may write (A) as follows:

$$(M_1 + M_2 + M_3)v^n = M_1v + M_2v^2 + M_3v^3. \qquad (B)$$

If there had been four debts, M_1, M_2, M_3, and M_4, due in 5, 7, 8, and 10 years, respectively, our equation of value would be written

$$(M_1 + M_2 + M_3 + M_4)v^n = M_1v^5 + M_2v^7 + M_3v^8 + M_4v^{10}, \quad (C)$$

in which the terms on the right would represent individually the present value of the four debts and the term on the left would represent the present value of the sum of the debts if paid n years hence.

In general, if there are debts, M_1, M_2, M_3, . . . , M_k due in n_1, n_2, n_3, . . . , n_k years, the fundamental idea of equivalence expressed in formulas (B) and (C) can be extended and we can write the *equation of value*,

$$(M_1 + M_2 + M_3 + \cdots + M_k)v^n = M_1v^{n_1} + M_2v^{n_2} + M_3v^{n_3} + \cdots + M_kv^{n_k}, \quad (1)$$

in which $v = \dfrac{1}{(1 + r)}$

Formula (1) is very useful for working many problems in finance where two or more obligations of unequal amount are to be exchanged for one or more amounts at other dates. Table IV (page 405) gives the values of v for certain values of n and r and makes possible a great reduction in the amount of arithmetic computations that would otherwise be required.

Now let us return to the example with which we began this discussion and employ formula (1) together with Table IV to solve our problem. We have

$$Sv^n = M_1v^{n_1} + M_2v^{n_2} + M_3v^{n_3},$$

in which $M_1 = 800, \quad M_2 = 600, \quad M_3 = 400, \quad S = 1{,}800,$

$n_1 = 1, \quad n_2 = 2, \quad n_3 = 3, \quad v = (1.02)^{-1},$

$n = ?$

Substitute,

$$1{,}800(1.02)^{-n} = 800(1.02)^{-1} + 600(1.02)^{-2} + 400(1.02)^{-3}.$$

Use Table IV,

$$\begin{aligned} 1{,}800(1.02)^{-n} &= 800(0.980392) + 600(0.961169) + 400(0.942322) \\ &= 784.3136 + 576.7014 + 376.9288 \\ &= 1{,}737.94. \end{aligned}$$

Take logarithms, and use Table I,

$$\begin{aligned} \log 1{,}800 - n \log (1.02) &= \log 1737.94, \\ 3.25527 - 0.00860n &= 3.24002, \\ -0.00860n &= -0.01525, \\ n &= 1.77. \end{aligned}$$

As indicated by this example, the use of tables usually facilitates the computations involved in many problems arising in business, finance, and statistical analyses.

PROBLEM

There are four debts of \$1,000, \$2,000, \$3,000, \$4,000 due in 3, 5, 7, and 9 years, respectively, without interest. If we assume that money is worth 1%, when may one single payment, equal to the sum of the several debts, be made so as to settle the obligations equitably?

Answer: 6.98 years.

The foregoing illustration and problem illustrate the use of an equation of value in a type of problem occurring frequently in commuting financial obligations. Our task in each was to find the date at which a series of debts due at different times may be equitably discharged by a single payment equal to the sum of all the debts. *Algebraically, our task was to solve formula* (1) *for* n *when all the other elements were known.*

A second problem calls for finding a single sum which may be paid at a certain date in exchange for a series of sums due at different times, when money is worth a given rate. *Here we must solve Formula* (1) *for* S.

Illustrations:

1. Three debts of \$100, \$200, and \$300 are due in 5, 7, and 10 years, respectively. What sum 3 years hence is sufficient to dis-

charge all three debts if money is worth 2%?

$$Sv^n = M_1v^{n_1} + M_2v^{n_2} + M_3v^{n_3}.$$

Substituting for the M's and n's,

$$Sv^3 = 100v^5 + 200v^7 + 300v^{10}.$$

Dividing by v^3,

$$S = 100v^2 + 200v^4 + 300v^7.$$

Substituting for the v's values found in the present value table (Table IV) in the 2% column,

$$\begin{aligned} S &= 100(0.961169) + 200(0.923845) + 300(0.870560) \\ &= 96.1169 + 184.769 + 261.168 \\ &= \$542.05. \end{aligned}$$

2. A man wishes to settle in one payment a debt of \$600 due in 3 years and another one of \$200 due in 7 years. Show the equation of value obtained if he wishes to pay (*a*) now, (*b*) in 3 years, (*c*) in 5 years, (*d*) in 12 years, if money is worth 2%.

Solution:

$$Sv^n = M_1v^{n_1} + M_2v^{n_2}, \qquad M_1 = 600, \qquad M_2 = 200,$$
$$n_1 = 3, \qquad n_2 = 7.$$

(*a*) Here, we have $n = 0$. Hence, since $v^0 = 1$,

$$S = 600v^3 + 200v^7,$$

or, since $v = (1 + r)^{-1}$,

$$S = 600(1.02)^{-3} + 200(1.02)^{-7}.$$

A convenient way of thinking of this relationship is to observe that since both debts are to be settled before they are due, each debt must be *discounted* (that is, we must find its *present value*), one for 3 and the other for 7 years.

(*b*) Here, we have $n = 3$. Hence,

$$\begin{aligned} Sv^3 &= 600v^3 + 200v^7, \\ S &= 600 + 200v^4 \\ &= 600 + 200(1.02)^{-4}. \end{aligned}$$

In this case the \$600 debt is due at the date of settlement and hence is neither *accumulated* nor *discounted*. Since the \$200 debt is to be paid 4 years before it is due, it must be discounted for that period of time.

(*c*) Here, we have $n = 5$. Hence,

$$\begin{aligned} Sv^5 &= 600v^3 + 200v^7, \\ S &= 600v^{-2} + 200v^2 \\ &= 600(1.02)^2 + 200(1.02)^{-2}. \end{aligned}$$

The \$600 debt is to be paid 2 years after it is due and thus must bear compound interest for that period of time. On the other hand, the \$200 debt must be discounted for 2 years since it is to be paid 2 years ahead of time.

(*d*) Here, we have $n = 12$. Hence,

$$\begin{aligned} Sv^{12} &= 600v^3 + 200v^7, \\ S &= 600v^{-9} + 200v^{-5} \\ &= 600(1.02)^9 + 200(1.02)^5. \end{aligned}$$

Both debts are to be paid after the dates due. The \$600 debt is at compound interest for 9 years, while the \$200 debt is accumulated for a period of 5 years.

PROBLEM

Given debts of \$320 and \$650 due in 2 and 6 years, respectively. If money is worth $1\frac{1}{2}\%$, obtain the equation of value for finding an equivalent single payment (*a*) now, (*b*) in 2 years, (*c*) in 4 years, (*d*) in 6 years, (*e*) in 8 years.

Answer:

(*a*) $S = 320(1.015)^{-2} + 650(1.015)^{-6}$,
(*b*) $S = 320 + 650(1.015)^{-4}$,
(*c*) $S = 320(1.015)^2 + 650(1.015)^{-2}$,
(*d*) $S = 320(1.015)^4 + 650$,
(*e*) $S = 320(1.015)^6 + 650(1.015)^2$.

Exercises

1. An electric calculating machine was purchased for \$500 cash and \$200 at the end of each year for 3 years. What is the cash value of the machine, if money is assumed to be worth 1%?

2. When may debts of \$250, \$340, and \$200 due in 3, 6, and 8 years, respectively, be equitably discharged by a single payment of \$800 if money is worth 2%?

3. In making a certain purchase a man was given two alternatives—he could pay either \$130 cash or \$45 down, \$45 in 6 months, and \$45 at the end of the year. If money is worth 2% compounded semiannually, which alternative is cheaper, and how much?

4. Find the "equated time" for the payment of \$900, \$1,400, and \$1,850 due in 5, 8, 12 years, respectively. Money is worth 1% per year.

5. Given the present value of a debt, \$495.60; the face of the debt, \$500; and money worth 2%. When will the debt be due?

6. Given a debt of \$1,500 due in $3\frac{1}{2}$ years and money worth 1%. What is the present value of the debt?

7. A merchant owes notes at his bank as follows: \$100 due in 30 days; \$300 due in 60 days; and \$500 due in 120 days. If money is worth $1\frac{1}{2}$%, at what future date may \$900 be paid so as to settle all three notes?

8. At what rate of interest, compounded semiannually will \$3,000 amount to \$6,000 in 20 years?

9. The Legal Loan Company charges a rate of $3\frac{1}{2}$% per month. This rate is equivalent to what rate per year?

10. Two debts of \$600 and \$400 are due in 3 and 5 years, respectively. If money is worth 3%, find the amount necessary to settle both debts (*a*) now, (*b*) in 3 years, (*c*) in 4 years, (*d*) in 8 years.

11. A government bond selling for \$42.50 matures at \$50 in 10 years. What rate of interest is earned if the money is compounded annually?

12. A manufacturing concern sets aside every year \$2,000 in a depreciation fund. If the fund accumulates at the rate of $1\frac{1}{4}$% annually, what amount will be in the fund at the end of 12 years? Assume the \$2,000 is put into the fund at the beginning of each year.

13. What cash price for a house is equivalent to \$3,000 down and \$70 a month for 15 years if money is worth 4% compounded monthly?

14. By making use of the formula for the sum of a geometric series, prove that

$$(1+r)^{-1} + (1+r)^{-2} + (1+r)^{-3} + \cdots + (1+r)^{-n} = \frac{1-(1+r)^{-n}}{r},$$

and hence show that the present value of \$1 paid at the end of each year for n years, where money is worth r, is given by $\dfrac{1-(1+r)^{-n}}{r}$.

15. By making payments of $100 a month, a loan of $3,000 made to purchase a car is to be repaid. How long will it take to repay the loan if money is worth 4% compounded monthly? [*Hint:* Make use of the formula of Problem 14.]

16. By making use of the formula for the sum of a geometric series, prove that

$$(1 + r)^{n-1} + (1 + r)^{n-2} + \cdots + (1 + r)^1 + 1 = \frac{(1 + r)^n - 1}{r},$$

and hence show that the compound amount of $1 paid at the end of each years for n years is given by $\frac{(1 + r)^n - 1}{r}$.

17. Use the formula of Problem 16 to find the compound amount of $50 paid at the end of each year for 4 years if money is worth 3%.

18. When various debts are due within a year, an approximate average due date may be arrived at as follows: Assume a focal date (say the last day of the month preceding the first item). Multiply the amount of each debt due by the number of days intervening between the focal date and the due date of the debt; then divide the sum of these several products by the sum of the debts. Count forward from the focal date the number of days obtained in the quotient. The result will be the average due date.

Several invoices mature as follows: Apr. 14, $260; May 24, $500; Aug. 19, $120; Sept. 5, $350. At what date may these items be paid in one amount without loss to either party?

19. A man owes debts as follows: $300 due in 2 months, $500 due in 3 months, and $450 due in 5 months. By the method of Problem 18, find the average due date.

20. A dealer has the choice of paying a debt in one of the following two ways: (*a*) He may pay $500 cash. (*b*) He may pay $100 now, $200 at the end of 2 years, and $300 at the end of 3 years. Which is his better proposition if money is assumed to be worth $1\frac{1}{2}$%? How much better?

21. A merchant has debts of $100, $300, and $600 due in 2 years, 3 years, and 7 years, respectively. At what future date would $1,000 equitably settle all three debts in one payment? (Assume money to be worth $1\frac{1}{4}$%.)

22. A debt of $100 bearing compound interest at the rate of 1% compounded semiannually is due at the end of 2 years 6 months. If money is worth 2%, what is the present value of the debt?

23. Mr. Adams owes Mr. Smith two debts: $450 due in 3 years with interest at 2% compounded semiannually, $845 due in 5 years with simple interest at 4%. Find (*a*) the value of each note at maturity, (*b*) the date on which $1,400 as a lump payment will pay both debts if money is worth $1\frac{1}{2}$%.

24. Last week, a typist was paid $60 for 5 six-hour days and 10 hr overtime. This week, she was paid $48 for 5 six-hour days and 4 hr overtime. How much does she receive for each six-hour day and for each hour overtime?

25. An automobile radiator holds 18 qt of a mixture of antifreeze and water, 75% being water. How much should a station attendant drain out and replace with antifreeze if he wishes to have a mixture containing 50% antifreeze?

26. A theater containing 1,000 seats had a sellout performance with gate receipts of $1,790.50. Orchestra seats sold for $2.20 and balcony seats for $1.15. How many seats of each kind did the theater contain?

27. Farmer Brown has 2 more sheep than pigs, 12 more pigs than cows, and 44 all together. How many of each animal does he own?

28. Solve for n:

(*a*) $350(1.06)^n = 557.85$, (*c*) $3^{2n-1} = 243$,

(*b*) $(1.06)^n = 2.5404$, (*d*) $1.342 = \dfrac{91.32}{(1.05)^n}$.

29. Solve for x and y:

(*a*) $\begin{cases} 4x - 3y = 33, \\ 3x + 4y = 81; \end{cases}$

(*b*) $\begin{cases} 3x - 2y = 1, \\ 5x + 3y = 17; \end{cases}$

(*c*) $\begin{cases} x - 3y = 135, \\ 2x + 4y = 320; \end{cases}$

(*d*) $\begin{cases} \dfrac{3}{2 - x} = \dfrac{15}{2y + 1}, \\ \dfrac{2x + 3}{2} = \dfrac{5y}{4}; \end{cases}$

(*e*) $\begin{cases} \dfrac{32}{x} - \dfrac{7}{y} = 1, \\ \dfrac{4}{x} + \dfrac{14}{y} = \dfrac{9}{4}. \end{cases}$

30. Solve for x, y, and z:

(*a*) $\begin{cases} 3x + 2y - 4z = 5, \\ x + y + 2z = -9, \\ 2x - y - z = 3; \end{cases}$

(*b*) $\begin{cases} 2x + y + z = 5, \\ 3x - 2y + 2z = -10, \\ x + 3y - 3z = 15. \end{cases}$

31. Debts of $100, $200, and $300 are due in 3, 5, and 9 years, respectively. If money is worth 2%, at what future date may $600 equitably settle all three debts?

32. Given debts of $670 and $360 due in 3 and 5 years, respectively. If money is worth 2%, what single payment will settle the debts equitably at the end of 3 years?

33. How much money would be required 7 years from now to discharge a sum of $1,200 due in 10 years without interest and $2,000 due in 5 years but bearing interest at 4% if money is worth $1\frac{1}{2}$%?

34. Years ago a man invested \$10,000 at 5% at the time of his marriage. He used the accumulated sum on the 20th and 25th wedding anniversaries to build homes. What sum did he use on the 20th anniversary for a town home so that he had an equal sum on the 25th anniversary for a country home?

35. Find the present value of an annuity of \$1,000 per year at the end of each year for 20 years if money is worth 4% compounded annually?

36. A house may be purchased by paying \$1,200 at the end of each year for 10 years. What is the cash value of the house if money is worth $1\frac{1}{4}$%?

37. The Jackson Equipment Company will sell a machine for \$5,000 cash. If money is worth 3% compounded annually, what should the payments be at the end of each year for 10 years if the machine is bought on the installment plan?

38. The Westwood Realty Company will sell a piece of property for \$8,530 cash. If the property is bought on the installment plan, the payments at the end of each year for 10 years must be \$1,000. What does the company consider money to be worth? [*Hint:* Estimate the answer directly from Table VI.]

39. The Stellar Instrument Company will sell a telescope for \$80,000 cash. If it is bought on the installment plan, the payments at the end of each year must be \$10,000. If the company considers money to be worth 2% compounded annually, how long will it take to pay for the telescope on the installment plan?

40. If \$10,000 is deposited on Jan 2 in a bank that pays 1% compounded annually, how much can be withdrawn at the end of each year for 12 years, first withdrawal to be made on Jan 2 one year after deposit?

Self-Test

Time: 50 Minutes

(Score 10 points for each problem correctly solved)

1. At the birth of a male heir \$1,000 is deposited in a savings bank paying $1\frac{1}{2}$% interest compounded semiannually. How old will the child be when the amount accumulates to \$1,200?

2. A man owes two debts. One is for \$2,000 due in 4 years, and the other is for \$1,000 due in 10 years. He falls heir to a sum of money and wants to pay the debts now. What lump sum would discharge the debts equitably if money is worth 2%?

3. On what date would \$2,800 equitably pay both debts in Problem 2?

4. A tobacco dealer wishes to mix two blends of tobacco, one costing \$1.80 per pound and the other costing \$1 per pound, to obtain a mixture of 25 lb which would cost \$1.50 per pound. How many pounds of each kind should he use?

5. Solve for x in the equation $7(3.02)^x = 65.1$.

6. Solve for x and y:

$$\begin{cases} 2x + 3y = 21, \\ 3x - 5y = -16. \end{cases}$$

7. Solve for T in the equation $T^2 = 5.5T - 7.5$.

8. Find the present value of an annuity of \$200 per year at the end of each year for 8 years if money is worth 1% compounded annually.

9. How many payments of \$100 at the end of each year must be made to amount to \$1,341, with interest at 2% compounded annually?

10. Solve $P = R\dfrac{1 - (1 + r)^{-n}}{r}$ for R.

Chapter X

FURTHER STATISTICAL METHODS

Introduction. In Chapter III the student was introduced to the basic principles underlying the gathering, classification, and simple description of data. In that chapter our attention was devoted to the computation of measures of central tendency, including the mean, the median, and the mode. The extent to which data tended to be scattered or dispersed about a central value was studied from the standpoint of average deviation, standard deviation, and point measures of dispersion. These measures of central tendency and dispersion are but a few of the many statistical devices that play an important part in the *description* of data compiled in the many business fields.

In the present chapter we shall devote our attention to a few basic concepts and statistical methods used in making *inferences* from data. Only the applied aspects of these methods will be emphasized, since an understanding of the statistical theory behind them presupposes a comprehensive understanding of mathematical theory.

In the modern use of statistics the investigator is usually interested in drawing conclusions (often of a tentative nature) from data resulting from a designed experiment. In other words, he plans how to gather his data, and what data to obtain, so that he may draw conclusions sufficiently reliable to aid in making some decision. For example, an advertiser makes a survey of potential viewers, and uses the information obtained to help in preparing a schedule for television programs. A railroad accounting office may take a sample of a relatively few waybills from a large number of them and obtain percentage figures that can be used in making distribution of revenue among appropriate recipients. The use of statistics for inferential purposes depends primarily upon the theory of probability, a subject that is receiving more and more attention in a great number of fields of application. For this reason, we shall consider a few of the basic ideas involved as preliminary to a few applications of this theory.

99. Combinations and Permutations. The study of probability requires a knowledge of the elementary mathematical theory of *combinations* and *permutations*. The word "combination" refers to a *group* of things without regard to their relative arrangement, whereas the word "permutation" pertains to *arrangement* of the members of a group. Thus, the combinations of the letters *a*, *b*, and *c* taken two at a time are *ab*, *ac*, and *bc*. The different permutations of these letters, taken two at a time, are *ab*, *ac*, *bc*, *cb*, *ca*, and *ba*.

The fundamental principle upon which the study of combinations and permutations is based may be stated as follows:

If one thing can be done in M ways and (after it has been done in any one of these ways) a second thing can be done in N ways, then the two together can be done in MN ways.

Illustration: A certain automobile loan company has 3 different inspectors and 4 district offices. In how many ways (1 inspector to an office) can the inspectors be assigned to offices?

Solution: The first inspector can be assigned to any one of the 4 offices. When 1 inspector has been assigned to any one of these offices, the second inspector can be assigned in any one of 3 ways; that is to say, the first two inspectors can be assigned in 4 × 3 ways. With any of these 12 ways, the third office can be "filled" in 2 ways. Hence, the number of different ways the 3 offices can be assigned inspectors is $4 \times 3 \times 2 = 24$.

The basic principle can be generalized so as to include a third, fourth, or fifth thing, and so forth; and general formulas for combinations and permutations can be derived. We state here several of the basic theorems:

Theorem I. *The number of permutations of n different things taken r at a time is*

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)(n-2) \cdots (n-r+1). \quad (1)$$

Corollary. The number of permutations of n different things taken n at a time is

$$P(n,n) = n!. \tag{2}$$

The symbol $n!$, called *n factorial*, represents the product of all the integers from 1 up to n inclusive and is defined to equal 1 when $n = 0$. Thus, $3!$ means $1 \times 2 \times 3 = 6$, and $5!$ means $1 \times 2 \times 3 \times 4 \times 5 = 120$.

Illustration: Find the number of permutations of 6 different things taken 4 at a time; taken all at a time.

In the first case, $n = 6$, and $r = 4$. Hence $P(6,4) = 6 \times 5 \times 4 \times 3 = 360$.

In the second case, $n = r = 6$. Hence, $P(6,6) = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

Theorem II. *If n things, in which n_1 are alike, n_2 others alike, n_3 others alike, and so on, are taken all at a time, then the number of permutations is*

$$\frac{n!}{n_1!n_2!n_3! \cdots}. \tag{3}$$

Illustration: How many different lighting arrangements can be made by using 3 red and 2 green lights?

Solution: We have $n = 5$, $n_1 = 3$, $n_2 = 2$. Hence the number is

$$\frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} = 10.$$

Theorem III. *The number of combinations of n different things taken r at a time is*

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$
$$= \frac{n(n-1)(n-2) \cdots (n-r+1)}{r!}. \tag{4}$$

Illustration: In how many ways can a debating team of 5 members be chosen from a public speaking class of 12?

Solution: We have $n = 12$ and $r = 5$. Hence,

$$C(12,5) = \frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5} = 792.$$

PROBLEMS

1. In how many different ways can the colors red, white, and blue be arranged, 3 at a time and no 2 alike?

2. In how many ways can 8 students arrange themselves in a row of 8 chairs?

3. How many different numbers of 3 digits each can be formed from the digits 3, 5, 7, 9?

4. In a metropolitan area, telephone numbers are formed by writing 2 letters followed by 4 digits. Allowing duplication of both letters and digits, how many different telephone numbers could be formed? (Assume no mechanical limitations.)

5. Find the value of $\frac{7!}{6!}$; $\frac{5!}{2!}$; $\frac{n!}{(n-1)!}$.

6. In a certain state, license plates have 2 letters followed by 3 digits. How many license plates can be issued, all different?

7. Signals on board ship are formed by placing 5 colored flags in a vertical column. How many signals are possible with 5 differently colored flags?

8. A theater party reserve a row of 8 seats at the theater, but only 4 of them attend. In how many ways can they occupy the seats?

9. How many more permutations can be made out of the letters "money" than out of the letters "trust"?

10. Jan Karl plays a game with 4 battleships, 3 cruisers, and 14 destroyers. In how many different ways can he arrange them in a single line by treating as identical those within each type?

11. A fruit dealer wishes to arrange a decorative line across the front of his show window, using 4 boxes of oranges, 3 boxes of apples, and 2 boxes of peaches. How many designs could be made?

12. The Ideal Soap Company wishes to give a commercial over a radio station every hour on the hour for 12 hr each day. The commercials are to consist of 5 advertisements for face soap, 3 for

laundry soap, and 4 for suds. In how many different ways can the programs be put on the air?

13. How many different arrangements can be made out of the word *Kokomo* taken all together?

14. A contractor plans to build 5 identical Colonial, 4 identical Cape Cod, and 2 identical ranch-style houses along one side of a city street. How many different arrangements are possible?

15. In a gambling concession, 10 nickels, 3 dollars, and 5 half dollars are to be arranged along the shelf by the roulette wheel. In how many ways can this be done?

16. A landscape gardener has 2 plants of each of 5 different kinds. In how many ways can he plant them along the front of a lawn? Around a circular flower bed?

17. In how many ways can a committee of 4 persons be appointed from a group of 10 men?

18. Two different colors are to be used in painting a certain make of automobile as it comes off the assembly line. How many paint jobs are possible if 5 different colors are available?

19. Out of an ordinary pack of 52 playing cards how many poker hands of 5 cards consisting of 3 hearts and 2 spades can be formed?

20. An entertainment bureau has under contract 8 tap dancers, 12 acrobats, and 20 singers. How many shows consisting of 4 tap dancers, 3 acrobats, and 14 singers can be formed?

21. On a mathematics examination consisting of 12 questions, the student is to answer any 10. How many different sets of answers are possible?

22. How many bridge hands of 13 cards can be dealt from an ordinary deck of 52 cards?

23. Out of an ordinary deck of 52 cards, how many different bridge hands containing 10 diamonds and 3 clubs can be dealt?

100. Probability. The theory of probability is said to have originated in the mathematical study of certain problems arising from games of chance. Following initial considerations of the subject during the seventeenth century, the concepts have subsequently been developed and applied in the field of insurance, the study of heredity, and a variety of other areas. Today, the theory of probability is regarded as fundamental in all the physical sciences and in the study of social phenomena. It is becoming very important in the field of business because of its relationship to the use of statistical methods in the analysis of collected data.

It is perhaps a matter for the pure mathematician to formulate a satisfactory definition of *probability*. As a basis for understanding how probabilities are calculated and how results can be interpreted, it is usually advantageous to distinguish between that kind of probability which is based upon logical conclusions and that which is determined by the examination of data. For example, when a coin is tossed, one may assume that it is as likely to fall heads as tails, so that the probability of a head or of a tail may be expressed accordingly. (In this thought there is a sort of circular definition, because "equally likely" assumes knowledge of the term "probability" that is being defined.) On the other hand, if the coin were imperfectly balanced, or if environmental conditions had complex effects upon the outcome of a toss, and if, say, 1,000 tosses of the coin showed 800 heads and 200 tails, it might seem to be better to estimate the probability of a head or a tail on the basis of this "lengthy experience." (In this thought, however, the definition of probability is made to depend upon a requirement of "long run," a condition that is itself difficult to define.) However, in spite of the difficulty of being able to state a precise definition of the term "probability," many important applications can be made of the concept. For the sake of terminology, we state that in those situations where logical possibilities are involved, we have *mathematical probability;* and when experience and observation are the determining features, we have *empirical probability*.

(*A*) *Mathematical Probability.* If a committee contains 4 men and 6 women and there is no system for determining in advance which person is selected—that is, if each person selected at *random* is equally likely to be selected—then the probability P that a member of the committee selected at random will be a man is

$$P = \frac{4}{4+6} = 0.4,$$

since a man can be selected in 4 of the 10 possible ways of making a selection. The "odds" are 6 to 4 in favor of selecting a woman, or 4 to 6 against the selection of a man. Accordingly, for *mathematical probability* we state the definition:

If an event can happen in any one of h ways and can fail to happen in any one of f ways where any of these h + f ways are equally likely to occur and only one can occur, then the probability P that the event will happen is defined to be

$$P = \frac{h}{h + f}, \tag{5}$$

and the probability Q that it will fail to happen is

$$Q = \frac{f}{h + f} = 1 - P. \tag{6}$$

Illustrations:

1. In running through a maze, a rat comes to a place where there are 7 alternate paths, 2 of which contain electric grids. If 1 of the 7 paths is taken, what is the probability that the rat will receive an electric shock?

Here, $h = 2$, and $h + f = 7$. Hence, $P = 2/7$.

2. Coupons are attached to 5% of the number of cakes of soap in a box. If the cakes are packed in a random fashion, what is the probability of selecting a cake with a coupon if one is drawn from a box containing 5 dozen cakes of soap?

Here, $h + f = 60$, $h = 3$ (5% of 60). Hence, $P = 3/60 = 1/20$.

In dealing with probability, we must be certain that all events are equally likely to happen and that selections or drawings are random. This is to be understood in the problems which follow.

PROBLEMS

1. A crate contains 12 boxes of black raspberries and 20 boxes of red raspberries. What is the probability that a box drawn out will be of the red variety?

2. A certain theater gives 25 prizes to holders of lucky tickets for a certain performance. If a total of 2,000 tickets have been sold, what is the probability that a holder of one of the tickets will win a prize?

3. At a party the names of 10 guests are placed in a hat. If 6 of the guests are men and 4 are women, what is the probability that a man whose name has been drawn out first will draw a woman's

name from those which remain? (Note that the man's name first drawn has not been replaced when the second drawing is made.)

4. In cutting for partners in a certain card game, the couple cutting the two high cards are to be partners. If Adams, Smith, and Jones cut a 6-spot, a 9-spot, and a jack, respectively, and if these cards are not replaced, what is the probability that the fourth player, Craig, will be Jones's partner? (The fourth player must cut a 10-spot or better.)

5. What is the probability that a single throw of a die will give a number above 4?

6. What is the probability that 1 card drawn from a deck of 52 cards will be an ace? A heart? The king of hearts?

7. A roulette wheel has the numbers 1 to 50 on it, and the player wins if a number divisible by 3 comes up. What is the probability of winning?

8. What is the probability of tossing a head on 1 throw of a coin? Of throwing an ace with one throw of a die?

9. You will either know the next person you meet on the street, or you won't. Therefore, the probability that you will know the next person you meet is 1/2. True or false? Why?

10. In shooting at the side of a barn from 10 yd, you will either hit or miss; therefore, the probability of your missing the barn is 1/2. True or false? Why?

11. In playing bridge, neither your hand nor dummy contains the king of spades. What is the probability that the player to your left has it? (Neither opponent has bid.)

12. The board of directors of a going concern decide to have their monthly meeting the last week in the month. If they draw from a hat to decide on the day of the week, what is the probability that it will be Wednesday? That it will not be Sunday?

(*B*) *Empirical Probability*. Let us suppose that in a certain city 42,000 voters voted for a new library and 28,000 voted against it. What is the probability that a voter chosen at random within the city would be in favor of the new library? In this situation, there is no mathematical way to state the desired probability without resorting to data at hand. Out of the total of 70,000 votes cast, 42,000 were for the library, so that we may say that the *relative fre-*

quency of voters in favor of the library is 42,000/70,000 = 0.60, or 60%. In the absence of any other information, we may say that the desired probability is 0.6. To accept this figure means essentially that, if samples of voters were questioned concerning the library issue, the ratio of voters in favor to the total number questioned would in some way approach 0.6 as more and more voters were questioned, or that sample ratios would tend to cluster about this ratio as more and more samples were taken. Accordingly, for *empirical probability* we state the definition:

If each trial of an event must result in either success or failure, and if after N trials there have been S successes and F failures, the relative frequency of success is S/N and the relative frequency of failure is F/N. If we assume that the probability of success in a single trial is equal to the relative frequency for an acceptably large value of N, the empirical probability P of success is defined to be

$$P = \frac{S}{S + F} = \frac{S}{N}, \tag{7}$$

and the empirical probability Q of failure is

$$Q = \frac{F}{S + F} = \frac{N - S}{N} = 1 - P. \tag{8}$$

When the mathematical definition of probability is extended to include probability based on data, care must be exercised in interpreting and using the obtained relative frequency. For example, if it is known that, out of 2,341 employees in a shop, 47 of them had suffered an accident during the course of a year, the empirical probability 47/2,341 = 0.02 would be of doubtful value if applied to another population or group of employees or even to the same group of employees in other years. This is because the obtained probability neglects differences in accident proneness, working conditions, and other relevant factors. Implications of this type are, in fact, recognized by certain insurance companies, as is evidenced by their provision of special rates for certain classes of individuals.

Illustration: Mortality Table. Table VII on page 413 is taken from the 1941 Commissioners Standard Ordinary Mortality Table and is the record of the number of deaths at different ages of a large number of persons. It is actually a distribution of the number of survivals out of an initial group of 1,023,102 people over a period of 99 years and serves as a basis for calculating empirical probabilities of life or death needed in the insurance field.

The mortality table consists of four columns as follows: The first gives the ages from 0 years to 99 years and is headed x; the second, headed l_x, gives the number living at the beginning of each age x; the third, headed d_x, gives the number dying between the ages x and $x + 1$; and the fourth, headed q_x, gives the yearly probability of dying. A portion of the table appears as follows:

Age x	*Number Living* l_x	*Number of Deaths* d_x	*Yearly Probability of Dying* q_x
18	955,942	2,199	0.00230
19	953,743	2,260	0.00237

In the second column we read that out of the original group of 1,023,102 persons, 955,942 survived to the age of 18 years and 953,743 survived to the age of 19 years. Hence, the probability that a person aged 18 will live to reach the age of 19 may be taken to be 953,743/955,942, or 0.99770, which means that about 998 persons out of 1,000 reaching the age of 18 may be expected on the basis of empirical probability to live to reach the age of 19. The probability that a person 18 will fail to reach the age of 19 is 1 − 0.99770, or 0.00230, the value shown opposite 18 in the q_x column.

PROBLEMS (Use Table VII)

1. Find the probability that a man aged 25 will reach his next birthday.

2. Find the probability that a man aged 21 will die before reaching the age of 60. What is the probability that he will live to be 60?

3. If l_x is the number of men living at age x and l_{x+1} is the number living at the age $x + 1$, what is the probability that a man living at age x will live to be $x + 1$?

4. Assuming that 939,197 individuals living at the age of 25 are all insured for $2,000 each, what would be the cost of the insurance per person for 1 year if interest and other charges are neglected?

5. What is the probability that a boy entering college at the age of 17 will live 4 years?

6. At what age does a man have about a 50-50 chance of living another year?

7. By the will of J. I. Nacht, aged 75, John Nacht is the beneficiary. If we assume that it is certain that John will live for 4 years, what is the probability of his gaining his inheritance in that time?

8. What is the probability that a person 25 years of age will live to be 65 years old?

9. What is the probability that a man married at the age of 21 will die in the year after his golden wedding anniversary?

10. What is the probability that a man aged 25 will die before reaching the retirement age of 65?

11. If a boy is 14 when entering high school, what is the probability that he will die sometime during his third year?

12. A clerk aged 31 must work for his company 5 years, and during the next following 5 years he gets a substantial promotion. What is the probability that he will die in this second 5-year period?

13. Compute your own probability of living to the age of 65.

14. For your own case, find the probability that you will die within the next 10 years.

101. Value of an Expectation. At a certain carnival a prize worth \$5 was offered to the holder of the winning number when a wheel containing 20 numbers was spun fairly. What would be a fair price to pay for one of the numbers?

Since each number is equally likely to win the prize and a person who possessed all the numbers *must* win, the money value of each ticket is 1/20 of \$5, or \$0.25. In other words, this would be a fair sum to pay for each number.

We might also say that a person holding one of the numbers might reasonably expect $1/20 \times \$5$ as the price to be paid for his number by any one who wished to buy it. In general, *if P is the probability that a person will receive a sum S, then the value of his expectation E is said to be PS, that is,*

$$E = PS.$$

Illustrations:

1. A person is to receive \$3 if he succeeds in drawing a spade at the first drawing from a deck of 52 cards. What is the value of his expectations?

There are 13 spades in the total of 52 cards, so that $P = 13/52$, or $1/4$. Hence, $E = 1/4 \times \$3 = \$0.75$.

2. A turkey worth \$8 is being raffled off. One hundred tickets selling for 10¢ each are sold. What is the monetary value of expectation of winning to a person who has bought 2 tickets? What would have been a reasonable price per ticket?

$$P = 2/100 = 1/50, \qquad S = \$8.$$

Hence,
$$E = PS = 1/50 \times \$8 = \$0.16.$$

Since the expected average return is 8¢ per ticket, this amount would be considered a reasonable price per ticket.

PROBLEMS

1. At the end of 10 years a man will receive \$5,000 if he is alive. At 2% simple interest, find the present value of his expectation if his probability of living is 0.7. (*Note:* First find the present value of the amount, and then find the expected value of this sum.)

2. At a gambling table a stake of \$200 is contingent upon throwing a spot less than 5 with a single die. What is the mathematical expectation of each player?

3. A man aged 70 specifies in his will that in the event of his death before reaching the age of 75 his grandson is to receive \$4,000 for the latter's college education. What is the expected value of this sum?

4. What is the expectation of a person who is to receive a quarter if he throws a head with a penny?

5. What is the expectation of a person who is to get a dollar if he draws a heart from an ordinary deck?

6. What is the expectation of a person who is to get a dollar if he draws a jack or better from an ordinary deck?

7. A young man aged 20 on entering college is promised \$500 at the end of 4 years. At 4% simple interest, find the present value of his expectation.

8. A man aged 50 has a paid-up insurance policy which returns him $2,000 at the age of 65. If money is worth 3% simple interest, what is the present value of his expectation?

9. A college student aged 18 is to receive $1,000 at the end of 4 years. What is the value of his expectation?

10. For a boy aged 12 years, $1,000 is placed in trust at 2% simple interest. At the age of 18 the boy enters college. If the money were to be given him upon entering college, what is the value of his expectation at age 12? Assume money is worth $1\frac{1}{2}$%.

11. A punchboard contains 200 different numbers, 8 of which pay off $1 each if punched. What is a reasonable price to pay per punch?

12. A $5 prize is contingent upon throwing an even number with a die. What is a reasonable price to pay for the chance?

13. In a race with a purse of $10,000, the odds on a certain horse are 5 to 3. What is the mathematical expectation of the owners of the horse? (In race-track terminology this means the horse has 3 chances out of 8 to win.)

14. The cash drawer of a cafe has 3 bills each of $1, $5, $10, and $20 denomination. If the waitress picks a bill at random, what is her mathematical expectation? (*Note:* Add 4 separate expectations together.)

102. Laws of Probability. Certain probability theorems are based upon the fundamental principle (see Section 100) and pertain to different kinds of events, depending on whether they are *independent*, *dependent*, or *mutually exclusive*. If the happening of one event does not affect the happening of another event, the two events are said to be *independent*. On the other hand, if the happening of one event is affected by the happening of another event, the two are said to be *dependent*. If the happening of one event excludes the possibility that another event will happen, the two events are said to be *mutually exclusive*.

In the cases of independent and of certain dependent events, we have what is sometimes called "both-and" probability. For example, suppose that the probability is 0.93 that a certain police officer will hit a given target under given conditions and that the corresponding probability for a new recruit is 0.24. Then, if we assume that the two events are independent, we may state that the probability that they will both hit the target is $P = (0.93)(0.24) = 0.22$. This example illustrates the following:

Theorem IV. *If $p_1, p_2, p_3, \ldots, p_r$ are the separate probabilities of r independent events, the probability that all these events will happen together at a given trial is the product of these separate probabilities, that is,*

$$P = p_1 \times p_2 \times p_3 \times \cdots \times p_r.$$

Illustration:

1. The names of 3 men and 5 women are placed in a hat. If replacements are made after each drawing, what is the probability that a man and a woman will be drawn together on 2 successive drawings?

The probability of drawing a man's name is 3/8, and after his name is replaced, the probability of drawing a woman's name is 5/8. Hence,

$$P = 3/8 \times 5/8 = 15/64.$$

PROBLEMS

1. If 3 coins are tossed, what is the probability of obtaining 3 heads?

2. The probability that a man will live 15 years is 1/6, and the probability that his wife will live 15 years is 1/5. What is the probability that 15 years hence (*a*) both will be alive, (*b*) only the wife will be living, (*c*) both will be dead?

3. A man and wife are 32 and 23 years old, respectively, when they marry. What is the probability that they will both live to celebrate their golden wedding anniversary?

4. What is the probability of throwing 2 aces in succession with 1 die?

5. From an ordinary deck of 52 playing cards, A, B, and C each draw 1 card, the card being put back after each drawing. What is the probability that they all draw a club?

6. One cooky jar contains 5 wafers and 3 doughnuts, and a second contains 4 wafers and 5 doughnuts. If 2 items are drawn at random from each jar by a small child, what is the probability that the child gets 4 wafers to eat?

7. A man aged 60 promises his grandson aged 11 that if he, the grandfather, dies before reaching the age of 67 the boy shall inherit \$5,000 at the end of the 7 years. What is the mathematical expectation of the boy?

8. June has 2 tickets in a 100-ticket lottery, and Jane has 1 ticket in a 50-ticket lottery. What is the probability that they both win?

As an example of dependent events, suppose in the manufacture of certain machine parts it is known from experience that 5% of the parts are defective, and we wish to determine the probability that a sorter will select on his first 2 random drawings (without replacement) 2 defective parts out of a typical lot containing 40 parts. In this case, if it can be assumed that 2 out of the 40 are defective, the probability of obtaining a defective part on the first drawing is 2/40; and after a defective part has been selected on a first drawing, there will remain 1 defective part in the remaining 39 parts. Hence, $P_1 = 2/40$, and $P_2 = 1/39$; and since both of these events must happen, we have $P = (2/40)(1/39) = 0.0013$. Thus, we see that if we know the probabilities of events happening in succession, the probability of their happening together is the product of these probabilities. Hence, we have the following:

Theorem V. *If p_1 is the probability of a first event, p_2 the probability of a second event after the first has happened, p_3 the probability of a third event after the first two have happened, and so on, through r events, then the probability that all these will happen in order is*

$$P = p_1 \times p_2 \times p_3 \times \cdots \times p_r.$$

Illustrations:

1. What is the probability of drawing 2 aces in succession from an ordinary playing pack of 52 cards?

$p_1 = 4/52$ (since there are 4 aces in the pack);
$p_2 = 3/51$ (after the first ace has been drawn, there are 51 cards).

Hence, $P = 4/52 \times 3/51 = 12/2{,}652 = 1/221$.

2. The Tasty Candy Company places a prize in 1 out of every dozen boxes of candy. What is the probability that a customer

who purchases 2 boxes from a newly opened carton of 5 dozen boxes will receive two prizes?

$$p_1 = 5/60, \qquad \text{and} \qquad p_2 = 4/59.$$

Hence, $P = 5/60 \times 4/59 = 20/3{,}540 = 1/177.$

Problems

1. In manufacturing certain machine parts it is known that 5% of the parts are defective. What is the probability that a sorter will select on his first 2 drawings 2 defective parts out of a lot of 200 parts?

2. In a national lottery the numbers 1 through 5,000 were thoroughly mixed in a fish bowl. What is the probability that the numbers 1 and 2 will be drawn first and second, respectively?

3. A box contains cards numbered from 1 to 5, inclusive. One number is drawn, and without replacing it a second drawing is made. What is the probability that both are odd numbers? What is the probability that both numbers are odd if the first card is replaced before the second drawing is made?

4. What is the probability of drawing 4 aces and the joker in any order in succession from a pack of 53 cards?

5. Four contract bridge players draw cards for the deal. What is the probability that, drawing in succession, they all draw a 10 or better?

6. A bag contains 3 white and 5 black balls. Three balls are drawn out in succession. What is the probability that the first two will be white and the third one black?

When the events are mutually exclusive, we have what is sometimes called "either-or" probability. For example, suppose in selecting at random a person from a group of 6 teachers composed of 4 men and 2 women *either* a man *or* a woman may be selected. That is, the drawing of one excludes the drawing of the other. Since the probability of drawing a man is 4/6 and the probability of drawing a woman is 2/6, and we know that the probability of drawing either a man or a woman is a certainty, we have that the probability of drawing either a man or a woman is $4/6 + 2/6$, or 1.

Again, suppose that we wish the probability of drawing either 2 men or 2 women in 2 successive selections. We have seen that the probability of selecting 2 men in succession is $4/6 \times 3/5$, or $2/5$. Likewise, we find the probability of selecting 2 women in succession to be $1/3 \times 1/5$, or $1/15$. Hence, the probability of selecting *either* 2 men in succession *or* 2 women in succession is $2/5 + 1/15$, or $7/15$. More generally, we may state the following:

Theorem VI. *If $p_1, p_2, \ldots, p_r$ are the separate probabilities of r mutually exclusive events, the probability that one of these events will happen in a single trial is the sum of the probabilities of the separate events, that is,*

$$P = p_1 + p_2 + \cdots + p_r.$$

Illustrations:

1. The probability that a man will live 12 years is $1/8$; the probability that his wife will live 12 years is $2/9$. What is the probability that either the man or the wife will live 12 years?

$$p_1 = 1/8 \times 7/9. \qquad p_2 = 2/9 \times 7/8.$$

Hence, $$P = 7/72 + 14/72 = 21/72, \text{ or } 7/24.$$

2. An envelope contains 3 one-cent stamps, 7 two-cent stamps, and 6 five-cent stamps. What is the probability of drawing either a two-cent stamp or a five-cent stamp at a single draw?

The total number of possible cases is 16; the number of favorable ways for drawing a two-cent stamp is 7, and the number for drawing a five-cent stamp is 6. Hence,

$$P = 7/16 + 6/16 = 13/16.$$

PROBLEMS

1. If 2 dice are thrown, what is the probability that exactly one of the faces will be a 6?

2. In a single throw of 1 die what is the probability of getting at least a 3?

3. The probability that one archery student will hit a target is $1/4$ and that another will hit it is $2/5$. What is the probability that the target will be hit if they shoot simultaneously?

4. In one throw of 2 dice, what is the probability of getting a 7 or 11?

5. Three coins are tossed. What is the probability of getting at least 2 heads?

6. A man aged 42 and his wife aged 40 agree to provide $10,000 for a junior college if either one of them lives 25 years and the other dies. What chance does the college have of getting the money?

7. What is the probability that at least 1 of a set of twins aged 21 will live for 40 years?

8. What is the probability of drawing at least 1 queen from an ordinary deck of 52 cards in 3 successive trials if no replacements are made after each drawing?

9. In drawing 3 names from a hat containing 5 men's names and 3 women's names, what is the probability that you will draw either 2 men and 1 woman or 3 men?

10. Two words in the telegram, "Arriving Tuesday. Order huge supply of refreshments. Let everyone burst with joy," must be deleted to get within the 10-word limit. If this is done at random, what is the probability that they will be either the first two or the last two?

103. Probability Distributions. If a pair of dice is tossed, the sum of the number of dots appearing on the top surfaces must be 2, 3, 4, . . . , or 12. A sum of 2 can be made in only *one* way, namely, with 1 dot on each die; but the sum of 3 can be made in *two* ways, namely, with 1 dot on one die combined with 2 dots on the other; the sum 4 can be made with "2 and 2," "1 and 3," and "3 and 1," for a total of 3 ways, and so on. In all, there are 36 different possible outcomes, all equally likely, arising out of the toss of two dice, but some of these outcomes are equivalent in that they result in the same sum. The possible outcomes can readily be determined by making use of the following table:

Face of Die

		1	2	3	4	5	6
Face of Die	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

We see, for example, that the sum of 7 can be obtained in six ways, and that 11 can be obtained in two ways. If we let X take on the values 2, 3, 4, . . . , 12, and let $f(X)$ represent the corresponding probabilities of 1/36, 2/36, 3/36, and so forth, we have what is called a *theoretical probability distribution:*

X	2	3	4	5	6	7	8	9	10	11	12
$f(X)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

If the probabilities are accumulated as we increase the value of X, we obtain what is called the *cumulative probability distribution:*

X	2	3	4	5	6	7	8	9	10	11	12
$F(X)$	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	36/36

We observe that the probability distribution is completely specified by $F(X)$ since, for example, if we know the probabilities of obtaining, say, at most $X = 7$ (that is, a 2, or a 3, or a 4, or a 5, or a 6, or a 7) and at most $X = 6$, then the probability that $X = 7$ is given by $F(7) - F(6) = 21/36 - 15/36 = 6/36$, the same value as given by $f(7)$.

Illustration: What is the probability of obtaining either an 8 or a 9 as a sum in tossing 2 dice?

Solution: We have from the probability distribution for the tossing of 2 dice

$$F(9) = 30/36 \qquad \text{and} \qquad F(7) = 21/36.$$

Hence, the required probability is $F(9) - F(7) = 9/36$.

---

104. The Binomial Distribution. If 2 coins are tossed, they can fall in 3 possible distinct ways—2 tails, 1 head and 1 tail, or 2 heads; and since a head and a tail are the same thing as a tail and a head, the probabilities are 1/4 for 0 heads, 2/4 for 1 head, and 1/4 for 2 heads. Thus, the probabilities of exactly 0, 1, or 2 heads may be represented by the successive terms of the binomial

$$(1/2 + 1/2)^2 = (1/2)^2 + 2(1/2)(1/2) + (1/2)^2$$
$$= 1/4 + 2/4 + 1/4.$$

If 3 coins are tossed, the probabilities of exactly 0, 1, 2, or 3 heads may be represented by the successive terms of the binomial

$$(1/2 + 1/2)^3 = (1/2)^3 + 3(1/2)^2(1/2) + 3(1/2)(1/2)^2 + (1/2)^3$$
$$= 1/8 + 3/8 + 3/8 + 1/8.$$

Figure 9 shows the probabilities of 0, 1, 2, 3, or 4 heads when 4 coins are tossed. The height of each rectangle equals the corresponding term of the expansion

$$(1/2 + 1/2)^4 = 1/16 + 4/16 + 6/16 + 4/16 + 1/16.$$

Since the base of each rectangle is one unit and the sum of the altitudes of all of the rectangles is 1, the total area of all rectangles is 1 and the area of a given rectangle represents the probability of a certain number of heads. Thus, the shaded portion represents the probability of 3 heads when 4 coins are tossed. If the areas of any two of the rectangles are added, the combined area would represent the probability of *either* one *or* the other of the corresponding number of heads. Thus, the probability of either 0 heads or 1 head (that is, at least 1 head) is seen to be 0.0625 + 0.25, or 0.3125.

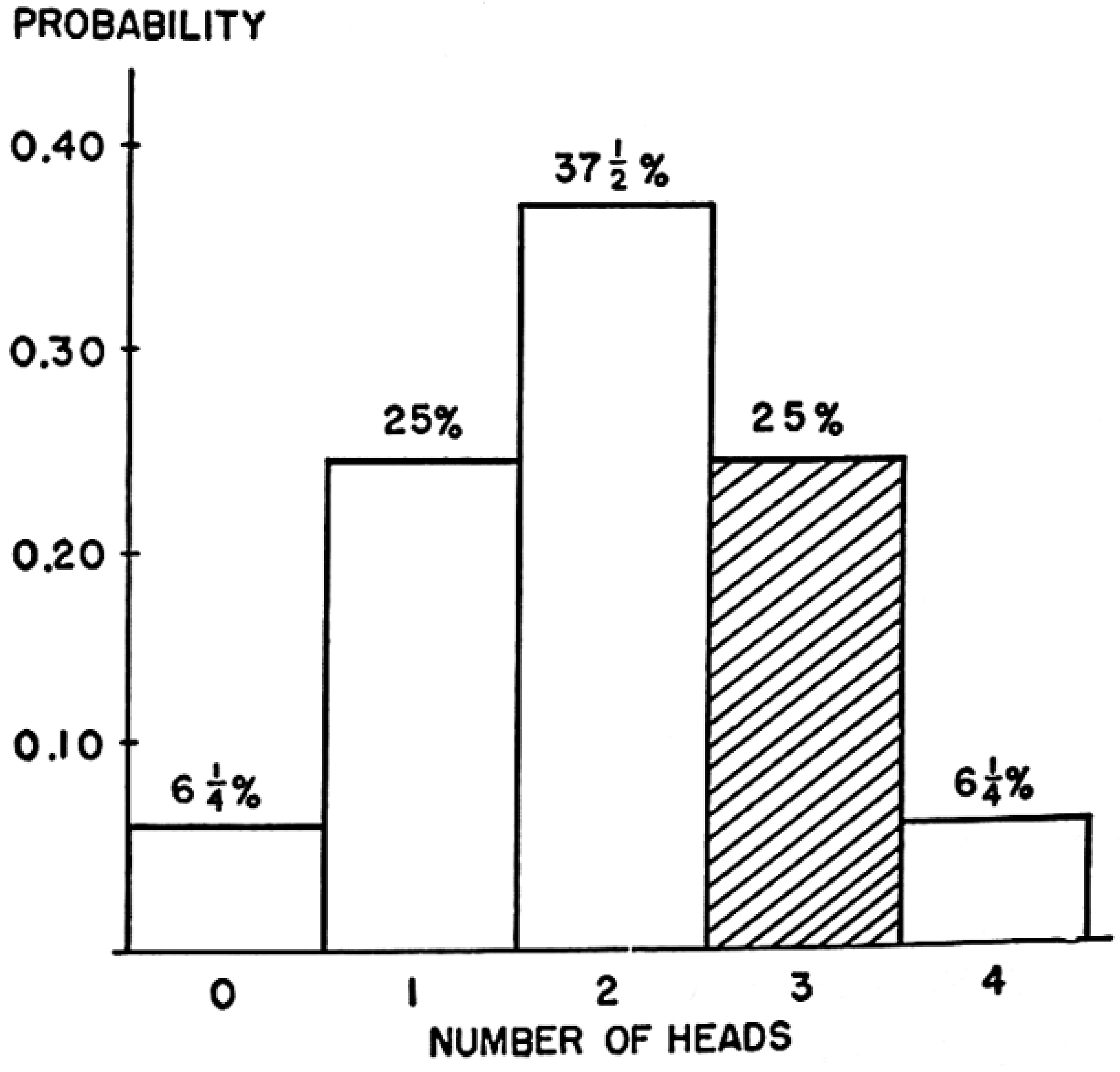

Figure 9. Probabilities of Heads when Four Coins are Tossed.

If a coin is tossed 10 times (or if 10 coins are tossed 1 time), the probabilities of 0 heads, 1 head, 2 heads, and so on, are given by the successive terms of the expansion of the binomial $(1/2 + 1/2)^{10}$. These probabilities are plotted in Figure 10, and the corresponding points are joined by line segments completing the frequency polygon.

In general, if p is the probability that an event will happen and q is the probability that it will fail to happen $(q + p = 1)$, then the probabilities that it will happen exactly $0, 1, 2, \ldots, X, \ldots,$ n times in n trials are given by the successive terms of the binomial expansion (see Section 92)

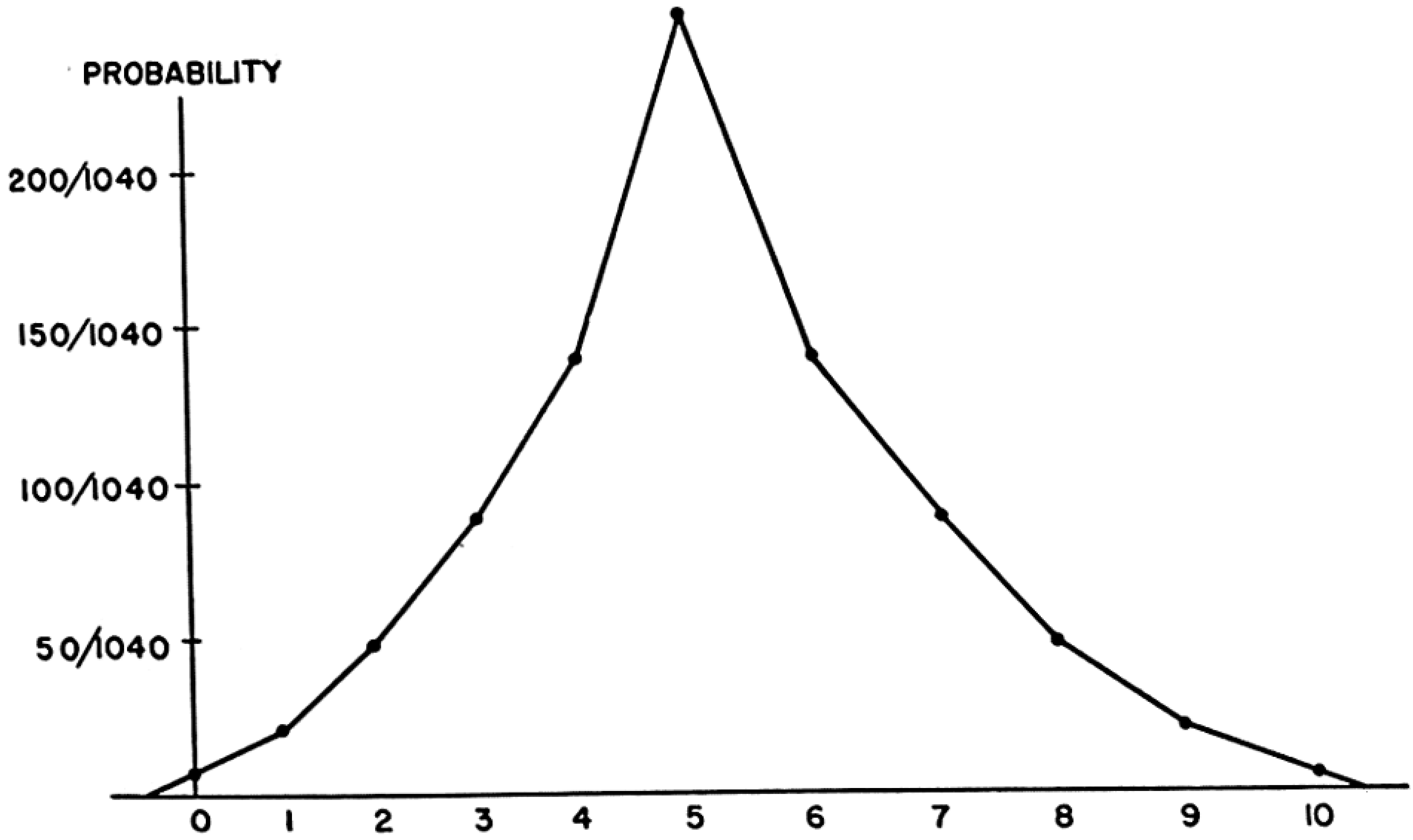

Figure 10. Probabilities of Heads when Ten Coins are Tossed.

$$(q + p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{1 \times 2} q^{n-2}p^2 + \cdots + p^n.$$

Furthermore, it can be shown that the general term of this expansion is

$$\frac{n!}{X!(n-X)!} q^{n-X}p^X,$$

and that, in the symbolism of combinations, this term can be written in the condensed form $C(n,X)q^{n-X}p^X$.

If we think of the number of successes, namely, $0, 1, 2, \ldots, X, \ldots, n$, as values taken on by a variable X and of the probabilities as relative frequencies of the several X values, we may write the *binomial distribution* in the form

$$f = C(n,X)q^{n-X}p^{X}. \tag{9}$$

The value of the mean of the binomial distribution may now be found in the usual way. We refer to the following table:

X	f	fX
0	q^n	0
1	$nq^{n-1}p$	$nq^{n-1}p$
2	$\frac{n(n-1)}{1\cdot 2}q^{n-2}p^2$	$n(n-1)q^{n-2}p^2$
3	$\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}q^{n-3}p^3$	$\frac{n(n-1)(n-2)}{1\cdot 2}q^{n-3}p^3$
. . .	. . .	. . .
$(n-1)$	nqp^{n-1}	$n(n-1)qp^{n-1}$
n	p^n	np^n
Totals	$(q+p)^n = 1$	$np(q+p)^{n-1} = np$

The total of the third column, ΣfX, may be recognized after the common factor np is taken out of every term. Since the mean, M, of the values of X is defined to be

$$M = \frac{\Sigma fX}{\Sigma f},$$

we have for the mean of the binomial distribution

$$M = np. \tag{10}$$

In a somewhat similar manner it may be shown that the standard deviation, σ, of the binomial distribution is given by the formula

$$\sigma = \sqrt{npq}. \tag{11}$$

Thus, for the distribution of 10 coins, in which $n = 10$ and $q = p = 1/2$, we have

$$M = 10 \times 1/2 = 5,$$

for the mean or *expected number* of heads in one throwing of 10 coins; and

$$\sigma = \sqrt{10 \times 1/2 \times 1/2} = 1.58 \text{ (approximately)},$$

for the standard deviation of the distribution of heads.

As has been indicated, the distribution $(q + p)^n$ will be symmetrical with respect to the mean np when $q = p$. If q is not equal to p, the distribution will be skewed to one side or the other. This skewness, or departure from symmetry, may be seen, for example, when the terms of the expansion $(1/4 + 3/4)^4 = 1/256 + 12/256 + 54/256 + 108/256 + 81/256$ are shown on a graph as in Figure 11.

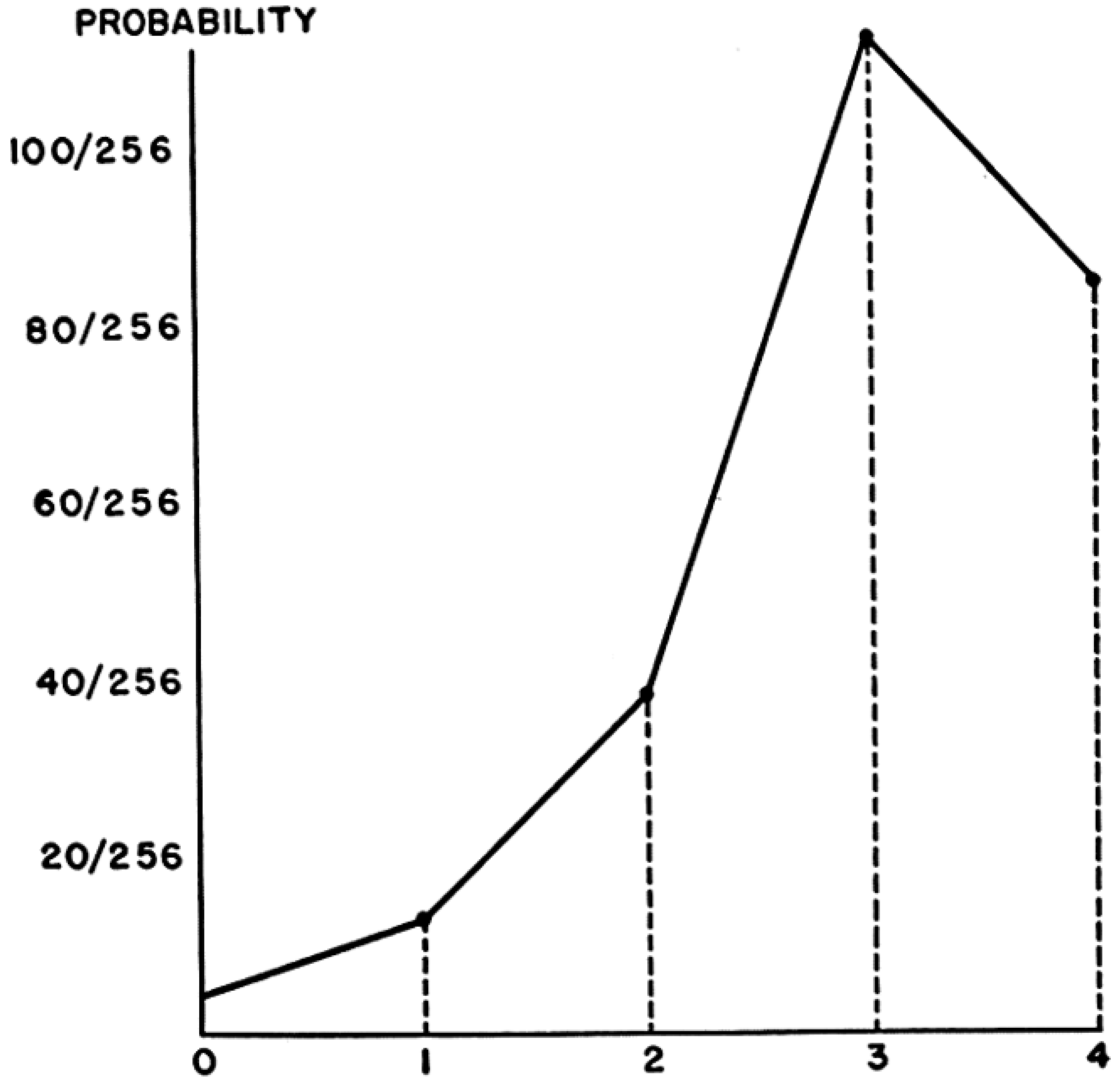

Figure 11. The Binomial Distribution $(\frac{1}{4} + \frac{3}{4})^4$

PROBLEMS

1. Figure 9 shows the probabilities for various combinations of heads and tails when 4 coins are tossed. What would be the probability of tossing (*a*) at least 3 heads? (*b*) At least 2 heads? (*c*) Neither all heads nor all tails?

2. Expand $(H + T)^n$, and show that the expansion may be written in the form

$$(H + T)^n = H^n + C(n,1)H^{n-1}T + C(n,2)H^{n-2}T^2 + C(n,3)H^{n-3}T^3 + \cdots .$$

3. Plot the histogram and the frequency polygon for each of the following binomial distributions. In each case find the mean and the standard deviation.

(*a*) $(5/6 + 1/6)^6$, (*c*) $(1/2 + 1/2)^6$,
(*b*) $(1/10 + 9/10)^4$, (*d*) $(3/4 + 1/4)^7$.

4. It is known that on the average 1 telephone number out of 4 called during the "peak" hour in a certain town is busy. If 5 randomly selected calls are chosen during this hour, what is the probability that not more than 1 of them will be busy? That all calls will be successfully completed?

5. The records of a certain hospital over a long period of time show that, in 60% of cases of a certain disease, the patient fails to recover. What is the probability of 1 death out of a group of 8 patients suffering from the disease? Explain how this probability might be of value in testing the effectiveness of a new drug.

6. The captain of a certain college rifle team has a long-time record of 1 bull's-eye in 6 shots at the target. What is the probability that he will hit the target at least once out of 4 shots?

7. A salesman has discovered that 10% of his calls on customers turn out to be successful. What is the probability that he will have more than 1 success during a morning in which he makes 5 calls? What is the probability that he will earn at least $40 if each successful call during the morning yields $10 profit?

8. In the manufacture of a certain radio tube it is known that 5% of the tubes are defective and must be discarded. What is the probability that a random sample of 10 tubes will show not more than 2 defectives?

105. The Normal Distribution. As *n* increases, the computation of the various probabilities given by the binomial distribution becomes very tedious. For example, to find the probability of exactly 50 heads in tossing a coin 1,000 times, it would be necessary to evaluate the expression

$$\frac{1{,}000!}{50!950!}\left(\frac{1}{2}\right)^{950}\left(\frac{1}{2}\right)^{50}.$$

In practice, we are usually content with a satisfactory approximation involving far less work.

If we plot the terms of the binomial expansion of $(1/2 + 1/2)^n$ as ordinates against the values of deviations from the mid-point of the distribution and draw, on a fixed base, the upper boundary of the corresponding histogram, it appears to approach a smooth curve as n is taken larger and larger. In Figure 12 this behavior is shown for $n = 25$ and $n = 100$.

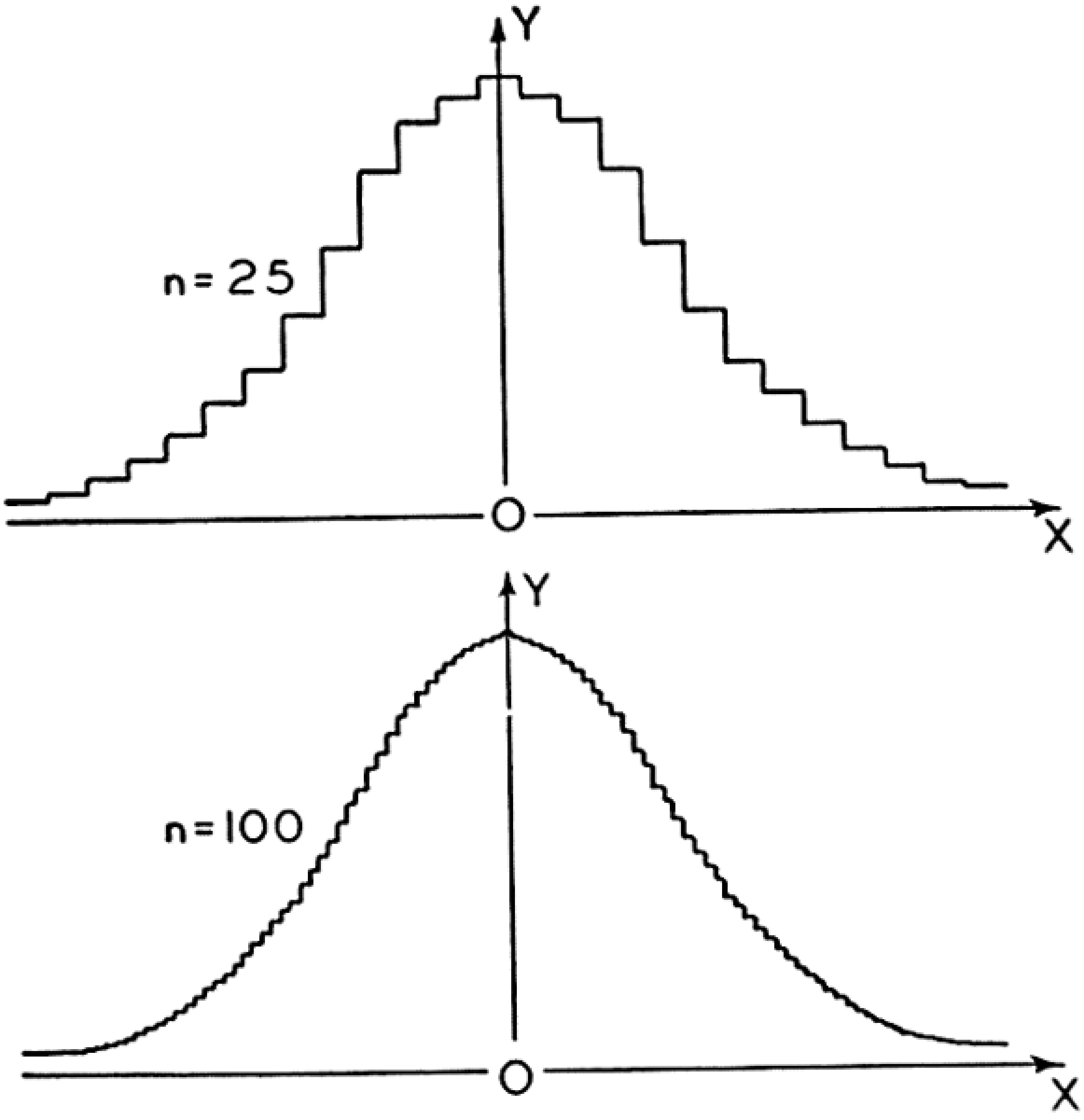

Figure 12. Upper Boundaries of Histograms for $(\frac{1}{2} + \frac{1}{2})^{25}$ and $(\frac{1}{2} + \frac{1}{2})^{100}$.

Figure 12 suggests the possibility of approximating the binomial by a bell-shaped curve. In mathematical statistics it is proved that, under certain conditions, the binomial distribution with $p = q = 1/2$ approaches the well-known *normal probability curve* when n is increased indefinitely, and the common base length of the intervals is decreased in the right manner. However, the derivation of the equation of the normal curve is not within the scope of this book. Usually the equation is written in the form

$$y = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2(x/\sigma)^2}, \tag{12}$$

in which x refers to a deviation from the mean, σ is the standard deviation, e is the constant $2.718 \cdots$, π is $3.141, \cdots$, and y

is called the probability density. This equation defines what is variously known as the *probability curve*, the *normal curve of error*, or the *normal frequency curve*. It has many applications in theoretical and applied statistics.

Equation (12) is actually a family of curves having the parameter σ. For different sets of measurements the curve will be found to vary from the low, flat form (see Fig. 13) to the high, narrow form. If σ is large (showing much dispersion about the mean), the curve takes the form B; if σ is small (showing clustering of the measurements about the mean), the curve takes the form A. This is a way of saying that the *precision* of a set of measures is inversely proportional to the value of σ.

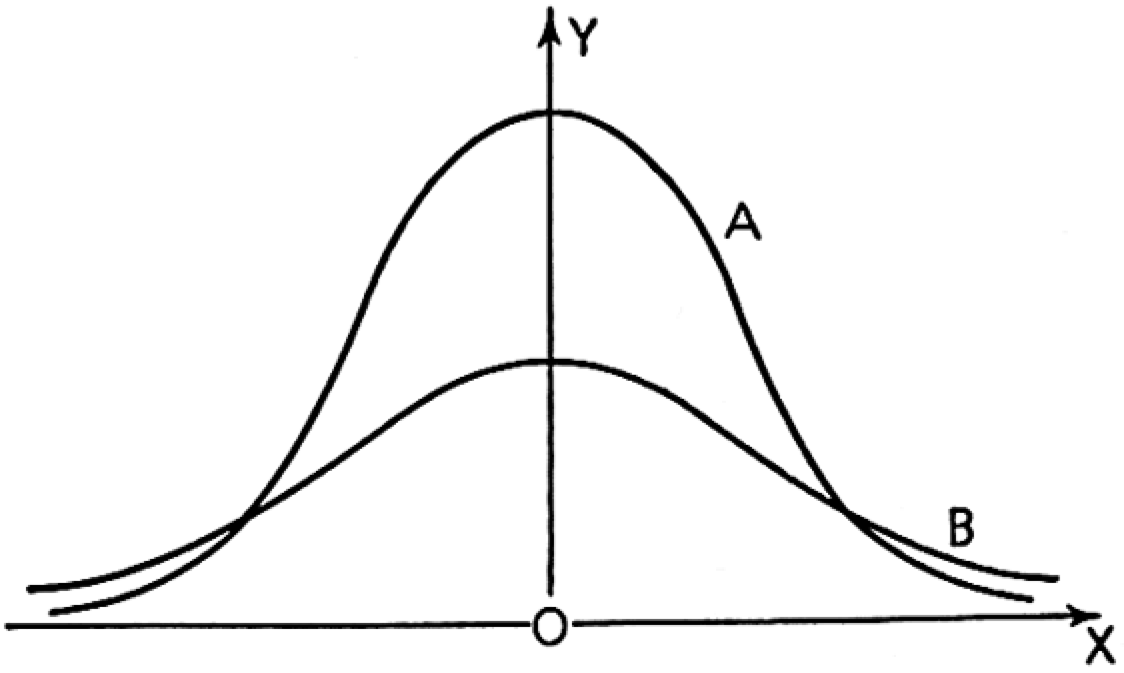

Figure 13. Normal Curves with Different σ's.

The equation of the normal curve is of the form

$$y = Ke^{-Cx^2},$$

in which K and C are positive constants whose values depend upon σ. In this form many of the important properties of the curve may be observed:

Symmetry. Since x may be replaced by $-x$ without changing the equation, the curve is *symmetrical* with respect to the y axis.

Intercept. The *y intercept* is the quantity K, since $y = K$ when $x = 0$. For values of x other than 0, y is less than K; hence, K is the maximum ordinate.

Asymptote. As x increases in numerical value, y approaches 0 as a limit, and the curve approaches the x axis as an *asymptote*.

The normal curve becomes very useful if K is selected so that the area under the curve equals unity. This permits us to interpret portions of the area as probabilities as in the case of the histogram for the binomial distribution. Convenient tables have been made available so that when we know how far a value is from the mean, we can, for example, tell what per cent of the distribution lies between that value and the mean; and when we know the per cent of measurements between the mean and a given value, we can find the deviation of that value from the mean. Such uses of the normal curve have many practical applications.

Equation (12) may be written in the form

$$y = \frac{1}{\sigma} f(t), \tag{12a}$$

in which $f(t)$ stands for the expression

$$\frac{1}{\sqrt{2\pi}} e^{-t^2/2},$$

where

$$t = \frac{x}{\sigma} = \frac{X - M}{\sigma},$$

and is called a *standard unit*. In Fig. 14 the percentages shown are the probabilities of the occurrence of deviations in terms of standard units between the indicated limits.

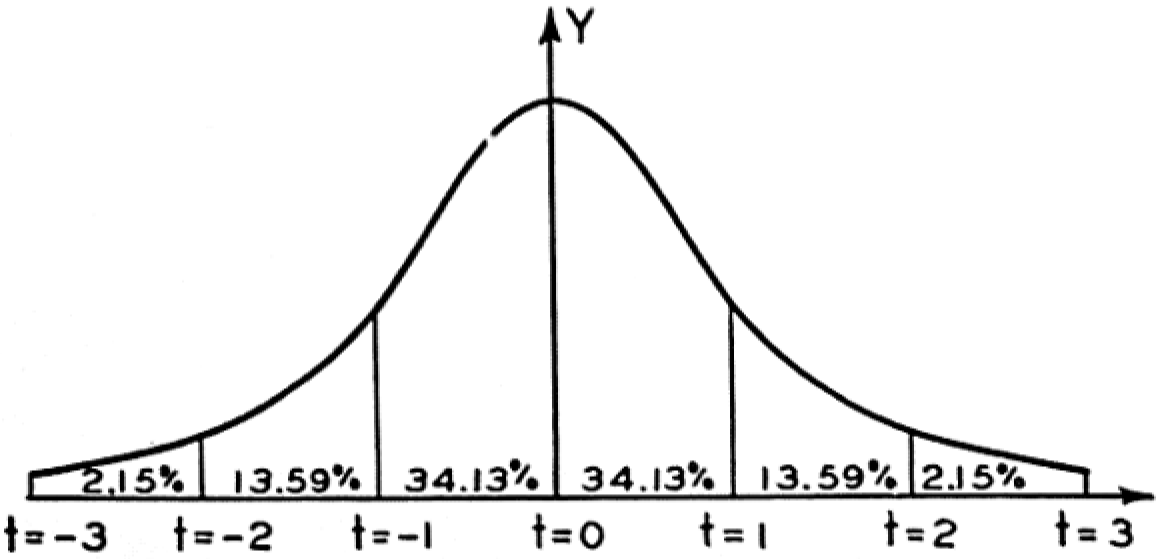

Figure 14. Normal Curve Percentage Distribution.

Table A is a portion of Table VIII in the Appendix and shows the values of $f(t)$ for different values of t, and also the area under the curve between $t = 0$ and these values of t. This table, known as a standard unit or t table, gives the per cent of a normal distribution lying between the mean and certain values of t up to 3. Since $f(t)$ is symmetrical with respect to t, it is not necessary to tabulate negative values of t. Also, since 49.87% of the measurements in a normal distribution lie within a range of 3 standard units on either side of the mean, it is seldom necessary to extend the table beyond $t = 3$. Table VIII is one of many extensive tables which statisticians use and which may be found in mathematical handbooks and statistical texts.

TABLE A

AREAS AND ORDINATES FOR THE NORMAL CURVE

t	$f(t)$	*Area*	t	$f(t)$	*Area*	t	$f(t)$	*Area*
0.00	0.3989	0.0000	1.00	0.2420	0.3413	2.00	0.0540	0.4772
0.10	0.3970	0.0398	1.10	0.2179	0.3643	2.10	0.0440	0.4821
0.20	0.3910	0.0793	1.20	0.1942	0.3849	2.20	0.0355	0.4861
0.30	0.3814	0.1179	1.30	0.1714	0.4032	2.30	0.0283	0.4893
0.40	0.3683	0.1554	1.40	0.1497	0.4192	2.40	0.0224	0.4918
0.50	0.3521	0.1915	1.50	0.1295	0.4332	2.50	0.0175	0.4938
0.60	0.3332	0.2257	1.60	0.1109	0.4452	2.60	0.0136	0.4953
0.70	0.3123	0.2580	1.70	0.0941	0.4554	2.70	0.0104	0.4965
0.80	0.2897	0.2881	1.80	0.0790	0.4641	2.80	0.0079	0.4974
0.90	0.2661	0.3159	1.90	0.0656	0.4713	2.90	0.0060	0.4981
						3.00	0.0044	0.4987

To illustrate the use of the normal curve, let us find the approximate probability of obtaining exactly 7 heads when 10 coins are thrown (see Section 104). The various probabilities are given by the terms of $(1/2 + 1/2)^{10}$; and the *exact* probability of obtaining 7 heads is given by $C(10,7)(1/2)^7(1/2)^3 = 0.117$. The *approximate* value of the probability may be found as follows: We have

$$M = np = 10(1/2) = 5,$$

$$\sigma = \sqrt{npq} = \sqrt{10(1/2(1/2)} = 1.581.$$

Since $X = 7$, this value represents a deviation of 2 units from the mean; hence, $x = 2$, and $t = 2/1.581 = 1.265$. In Table A, it is seen that $f(1.265)$ lies between 0.1942 and 0.1714. In more extensive tables, this value is found to be 0.1792. Then, since

$$y = \frac{1}{\sigma} f(t),$$

by Equation (12), we have

$$y = \frac{1}{1.581}(0.1792) = 0.113,$$

as the approximation to the exact value of 0.117.

As another illustration of the use of the normal curve, suppose that the scores on a nation-wide draft-deferment examination are distributed normally with a mean of 80 and a standard deviation of 5.0. What per cent of the scores could be expected to exceed 90? We have

$$X = 90, \qquad M = 80, \qquad \sigma = 5.0.$$

Hence,

$$t = \frac{X - M}{\sigma} = \frac{90 - 80}{5.0} = 2.0.$$

From the table, we see that 47.72% of the scores may be expected to lie between the mean and $t = 2.00$, which means that 50% −47.72%, or 2.28%, may be expected to lie beyond this point.

Table A shows that 25% of the normal distribution is included between the mean and a value of t somewhere between 0.60 and 0.70. This means that 50% of the distribution is included between that value of t laid off in either direction from the mean. By carrying the computation to more decimal places, it has been found that 50% of the normal distribution is included between $t = -0.6745$ and $t = +0.6745$.

Since $x = \sigma t$, it follows that 50% of a normal distribution lies between values of x corresponding to -0.6745σ and 0.6745σ. This numerical value of x is called *probable error*. It is that value of x which is just as likely to be exceeded as not. Graphically, x is the abscissa of that ordinate which bisects the area under the curve in either the first or second quadrant. Because of simplicity of calculation and usefulness in describing collected data, the probable error is a valuable tool for purposes of analysis. Applied to the draft-deferment scores, the probable error is $0.6745 \times 5 = 3.3725$, which means that 50% of the examination scores may be expected to lie within 3.3725 units on either side of the mean. That is, the limits for the middle 50% of the distribution of examination scores may be stated,

$$M \pm x = 80 \pm 3.37 = 76.63 \text{ and } 83.37.$$

PROBLEMS

1. Refer to Table VIII, and find the portion of the area under the normal curve between:

(*a*) $t = 0$ and $t = 1.00$,
(*b*) $t = 0$ and $t = 1.50$,
(*c*) $t = 0$ and $t = 2.00$,
(*d*) $t = 0$ and $t = 3.00$,
(*e*) $t = -1.00$ and $t = +1.00$,
(*f*) $t = -2.00$ and $t = +2.00$,
(*g*) $t = 0.50$ and $t = 1.20$,
(*h*) $t = -0.93$ and $t = +0.87$,
(*i*) $t = -1.45$ and $t = -0.15$,
(*j*) $t = 1.78$ and $t = 2.97$.

2. Ten coins are tossed upon a table. Use the normal curve and find the approximate probability of obtaining (*a*) exactly 4 heads; (*b*) exactly 6 heads; (*c*) either 4, 5, or 6 heads; (*d*) at least 3 heads.

3. For a distribution of daily sales it is known that the mean daily number of sales is 400 and the standard deviation is 55 sales. Under the assumption that the distribution is normal, find the probability of exceeding 500 sales on any given day.

4. For a normal distribution of measurements it is known that $M = 60$ and $\sigma = 4$. If measurements of 66 or greater are rejected as defective, what per cent of the measurements in a random sample can be expected to be rejected?

5. The mean for a distribution of 500 weights was found to be 160 lb, and the standard deviation was 25 lb. Assuming the distribution to be normal, about how many weights exceeded 200 lb?

6. The mean for 1,000 college grades in mathematics was 72, and the standard deviation was 12. What range of grades would probably include the middle 500 grades?

7. What is the probable error of 500 bowling scores whose mean and standard deviation were found to be 160 and 15, respectively? What interpretation would you give to this value?

8. Suppose that another 500 bowling scores had a mean of 160 and a standard deviation of 8. What conclusions would you draw in comparing these scores with those of Problem 7?

9. A director of consumer research obtained a sample of prices charged at 580 stores on a certain day for a standard make of a certain drug product, and obtained a mean of $0.87 and a standard deviation of $0.11.

(*a*) About how many stores probably were charging below $0.80?
(*b*) How many stores probably charged more than $0.95?
(*c*) About how many stores were charging more than $1?
(*d*) About half of the stores were charging prices within what range equally spaced above and below $0.87?

10. Draw and compare the graphs of the following probability curves:

$$\text{(a)}\quad y = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}. \qquad\qquad \text{(b)}\quad y = \frac{1}{2\sqrt{2\pi}}\,e^{-x^2/8}.$$

106. Sampling and Estimation. A great deal of statistical work depends upon the use of samples. Examples of situations in which information about a group, or *population*, of items is obtained by examining a portion, or *sample*, of the items may be found in wage studies, advertising surveys, time and motion studies, and many other phases of business activities.

If a very large number of samples could be drawn at *random* from a population of items, it would be found that they would tend to possess similar characteristics, but that they would not be identical. Observed differences would be present because of chance variation. This behavior can easily be demonstrated by considering the following "toy population" of only 4 numbers.*

Population: 2, 3, 5, 10.
Characteristic: Mean = 5.
Sampling procedure: Select a random sample of size $n = 3$.

* In practice, a population will usually consist of a relatively large number of items, and the sample size will ordinarily be comparatively small. The small population considered above is purposely chosen so that basic principles may readily be demonstrated.

For the purpose of seeing the possible results of estimating the mean of the population (known here to be 5) by the use of the sampling procedure described, let us write down all the possible samples (the sampling distribution) of size 3, and let us observe the information obtained.

Possible Samples	*Sample Mean*	*Probability of Sample*
2, 3, 5	$3\frac{1}{3}$	1/4
2, 3, 10	5	1/4
2, 5, 10	$5\frac{2}{3}$	1/4
3, 5, 10	6	1/4

We see that our *estimate* of the population mean varies from sample to sample and that each result is equally likely. The "expected value," or the average of these estimates is $\frac{1}{4}(3\frac{1}{3} + 5 + 5\frac{2}{3} + 6) = \frac{1}{4}(20) = 5$. Since the expected value equals the true, or population, mean, the sampling procedure, and consequently each estimate, is said to be *unbiased.* Although in this situation, involving so few numbers, sampling would not ordinarily be employed, it is interesting to observe that it is needless to consider all the items in the population if the required *accuracy* of our estimate need not be any closer than 2 units from the true mean of 5. It is because of the relationship of the saving in time and cost to the required accuracy and precision of an estimate based on a sample that sampling procedures are becoming so important in the business and industrial world.

A single sample is usually merely one of a large number of possible samples that might be taken from a population. The probability that a sample mean will exactly equal the population mean is, of course, small; but the probability that it will fall within a stated range depends upon the proportion of all possible means that fall within this stated range. For example, in the case of the foregoing illustration, we can say that the probability that a sample estimate will fall within 1 unit of the true mean is 0.50 (since 2 of the 4 possible means, namely, 5 and $5\frac{2}{3}$, are within this range). Similar probability statements can be made about other measures, such as the median, the standard deviation, and so forth.

The amount of fluctuation, or scatter, in a sampling distribution of a given measure, such as a mean, or a median, or a proportion, is usually measured by its *standard error,* that is, the standard deviation of the sampling distribution of that measure. The discussion in the preceding section dealing with the normal curve indicated that approximately 68% of the area under a normal curve lies between the limits $t = -1$ and $t = +1$, that is, within 1 standard unit on either side of the mean. This is true for a normal distribution of any measure and, in particular, for the distribution of sample means obtained from a normal distribution of items. Hence, $M \pm$ the standard deviation of the sample means can be expected to include 68% of the sampling distribution of means. Likewise, $M \pm$ two standard deviations of the sample means can be expected to include 95% of the sample means; and $M \pm$ three standard deviations of the sample means can be expected to include more than 99% of the sample means. This suggests that if we have a knowledge of the scatter of all possible sample means, then we are in a position to make a probability statement about obtaining one sample mean of some specified value. In practice, we do not, of course, know the nature of the theoretical sampling distribution of all possible samples; but, fortunately, statistical techniques have been devised by which certain characteristics of a population can be reliably inferred from those found in a single sample. Then, by the use of available formulas and tables, one can test the *significance* of a sample result.

It is not possible in these few introductory remarks concerning the many important sampling techniques that have been developed in recent years to give a complete explanation of the use of sampling in practical applications. However, we shall give here a few illustrations, together with the necessary formulas, of the use of certain standard sampling procedures in problems of a type encountered in business. In all cases, it will be assumed that the sample has been drawn in a random fashion from a normal population.

For use in the illustrations, we state without proof two standard error formulas commonly used:

$$\sigma_{\bar{X}} = \frac{\sigma_X}{\sqrt{n}} \qquad \text{(standard error of sample mean),} \tag{13}$$

in which σ_X is the standard deviation of the population, $\sigma_{\bar{X}}$ is the standard deviation of the mean, and n is the sample size;

$$\sigma_p = \sqrt{\frac{pq}{n}} \qquad \text{(standard error of per cent of occurrence of an event),} \tag{14}$$

in which p refers to the population proportion, $q = 1 - p$, and n is the sample size. In using these formulas, it is to be understood that the substitutions of quantities obtained from a single sample give only approximate results. However, experience has shown that for rather large samples the approximation is quite good. The student should not interpret the following discussion as being complete in any sense. The role of sampling in present-day business problems is becoming so important that additional study in this area is very strongly recommended.

Illustration 1: From records kept by a certain air line it has been found that for the population of men traveling in a certain type of airplane the mean gross weight is 160 lb and the standard deviation is 8 lb. What is the probability that a load of 36 men would average more than 164 lb per man?

Solution: Assumptions.

Distribution of weights is normal.
Population mean = 160 lb.
Population standard deviation = 8 lb.

We substitute $\sigma = 8$ and $n = 36$ in formula (13) and obtain

$$\sigma_{\bar{x}} = \frac{8}{\sqrt{36}} = \frac{4}{3}.$$

The difference between the sample mean and the theoretical or population mean is

$$164 - 160.$$

This difference divided by the standard deviation of the sample means (that is, standard error of the mean) indicates the number of standard units the mean in question (observed mean) deviates from the population mean. That is,

$$t = \frac{164 - 160}{4/3} = 3.$$

In Table VIII we find that the area under the normal probability curve between $t = 0$ and $t = 3$ is 0.4987. This means that the probability of exceeding this value is less than 0.01, or that one should expect a load of 36 men to average more than 164 lb in fewer than 1% of such samples.

Illustration 2: A group of television entertainers claimed that 25% of the families owning television sets in the local area were regular viewers of their program. To test this claim, a potential sponsor took a sample poll by telephone of 400 families owning television sets and found that 80 of these families were regular viewers. What decision could the potential sponsor make concerning the claim of the entertainers?

Solution: Assumptions.

Random sample of television owners.
Normal distribution of percentages of samples that are regular viewers.

The problem for the sponsor is to decide whether there is a *significant* difference between the 20% (80 out of 400) shown by the sample and the 25% (population value) claimed by the entertainers. This amounts to determining whether it can be considered "unusual" to obtain the sample result if the population value is truly 25%.

We substitute $p = 0.20$, $q = 0.80$, and $n = 400$ in formula (14) and obtain

$$\sigma_p = \sqrt{\frac{0.20 \times 0.80}{400}} = 0.02.$$

The difference between the sample per cent and the per cent claimed is

$$0.20 - 0.25.$$

This difference divided by the standard error of the per cent of occurrence is

$$t = \frac{0.20 - 0.25}{0.02} = \frac{-0.05}{0.02} = -2.5.$$

Using $t = 2.5$ in Table VIII (we are concerned only with the numerical value of t), we find that the probability of obtaining so large a value is 0.5000 − 0.4938, or 0.0062. This means that the probability is only about 6 in 1,000 that the claim of the entertainers is correct. If the sponsor had been interested in the probability of a 5% deviation in *either* direction from the 25% claim, the probability obtained would be doubled, giving a probability of 12 in 1,000.

The obtained probability of 0.0062 is so small that the sponsor would do well to conclude that the obtained 5% difference is *significantly* large. In other words, *if* the claim of the entertainers is in fact correct, then the sample result must be explained on the basis of its being an extremely unusual event.

PROBLEMS

In each of the following problems, assume that differences are significant if the value of t is as great as or greater than 2.0:

1. From long experience it has been established that the mean breaking strength of a certain type of wire is 80 lb and that the standard deviation is 16 lb. A random sample of 144 pieces of wire obtained from a large supply showed a mean breaking strength of 78 lb. Under the assumption that the breaking strengths are normally distributed, what is the probability that the sample mean will be 2 or more lb under the population mean? Would one conclude that the supply sampled is below standard?

2. A student survey of 400 automobiles in a college parking lot showed that 30% were of a gay color. Data obtained from official sources showed that 25% of the cars registered in the town were of the same gay color. Is the difference sufficiently large to indicate student preference for the gay color?

3. A distribution of invoices is assumed to be normal with a mean of $10.20 and standard deviation of $4.10. An invoice greater than $22 was found in a random selection. What is the probability that this would have happened if the assumptions were correct?

4. In a machine shop it was found that a certain operation required on the average 30 sec, with a standard deviation of 15 sec. A group of 16 workers were tested on the operation and averaged 24 sec. Would you conclude that this group was exceptionally fast on this operation? Why?

5. A shirt company sells shirts by mail and receives orders from an average of 10% of the letters sent out. A new approach is used in 625 letters, and from them 80 sales result. Would you conclude that the new approach is more effective than the old?

Self-Test

Time: 50 Minutes

(Score 10 points for each problem correctly solved)

1. The flags of France, Russia, England, the Netherlands, and the United States are to be hung in a row at a United Nations conference. In how many ways can this be done?

2. Bob Smith has 20 business books. He is going on a vacation and wishes to pick 3 books to take along. How many choices does he have?

3. What is the probability that a chimpanzee striking in succession 3 keys of a typewriter with 47 keys will type out the word "ape"?

4. Find the probability that a boy aged 10 years will live long enough to be eligible for election as President of the United States. (The President must be 35.)

5. In a national trapshoot extending over a period of years, Smith averages 97 clay pigeons out of 100, and Jones averages 98. What is the probability that at least 1 of the 2 men hits a clay pigeon when they shoot at the same time?

6. In 1 throw of 2 dice, what is the probability of getting a two or a twelve?

7. A box contains 1 red ball and 2 blue balls. If a single ball is drawn and replaced, what is the probability of obtaining 2 red balls in 4 such drawings?

8. The mean of a distribution of butter prices was found to be 72¢, and the standard deviation was 10¢. Of 100 prices sampled, how many were probably between 72¢ and 82¢?

9. The standard deviation of the weights of 1,000 freshman girls was found to be 11 lb. If the mean weight was 120 lb, what range of weights would probably include 97% of them?

10. The acceptable distribution of life of a certain light bulb has a mean of 800 hr and a standard deviation of 20 hr. A sample of 100 bulbs from a new type has a mean life of 790 hr. Should one conclude that the new type is below standard?

APPENDIX

TABLE PAGE

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Table I

COMMON LOGARITHMS OF NUMBERS

Note: The small tables included on the right side of the page under the heading of *P P* are tables of *Proportional Parts.* These small tables may be used as an aid in making interpolations quickly, for the values appearing in them are the same as those found by interpolation as discussed in the chapter on logarithms. To obtain log 10087, for example, we observe that the difference between the tabular values for log 1008 and log 1009 is 43. In one of the small tables under the heading 43 and opposite 7, the fifth digit in the number whose logarithm we wish to find, we read 30.1, which to the nearest integer is 30. This is the amount of correction to be added to 4.00346, the logarithm of 10080, to obtain log 10087. Hence, log 10087 = 4.00376. With a little practice, one can develop considerable facility in using the table of logarithms and a person with enough confidence will find that calculations can generally be made mentally.

Each mantissa marked with an asterisk (*) starts with the same two digits as the next row, rather than with the two digits at the beginning of its own row.

100–150

N	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
01	432	475	518	561	604	647	689	732	775	817
02	860	903	945	988	*030	*072	*115	*157	*199	*242
03	01 284	326	368	410	452	494	536	578	620	662
04	703	745	787	828	870	912	953	995	*036	*078
05	02 119	160	202	243	284	325	366	407	449	490
06	531	572	612	653	694	735	776	816	857	898
07	938	979	*019	*060	*100	*141	*181	*222	*262	*302
08	03 342	383	423	463	503	543	583	623	663	703
09	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
11	532	571	610	650	689	727	766	805	844	883
12	922	961	999	*038	*077	*115	*154	*192	*231	*269
13	05 308	346	385	423	461	500	538	576	614	652
14	690	729	767	805	843	881	918	956	994	*032
15	06 070	108	145	183	221	258	296	333	371	408
16	446	483	521	558	595	633	670	707	744	781
17	819	856	893	930	967	*004	*041	*078	*115	*151
18	07 188	225	262	298	335	372	408	445	482	518
19	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
21	08 279	314	350	386	422	458	493	529	565	600
22	636	672	707	743	778	814	849	884	920	955
23	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
24	09 342	377	412	447	482	517	552	587	621	656
25	691	726	760	795	830	864	899	934	968	*003
26	10 037	072	106	140	175	209	243	278	312	346
27	380	415	449	483	517	551	585	619	653	687
28	721	755	789	823	857	890	924	958	992	*025
29	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
31	727	760	793	826	860	893	926	959	992	*024
32	12 057	090	123	156	189	222	254	287	320	352
33	385	418	450	483	516	548	581	613	646	678
34	710	743	775	808	840	872	905	937	969	*001
35	13 033	066	098	130	162	194	226	258	290	322
36	354	386	418	450	481	513	545	577	609	640
37	672	704	735	767	799	830	862	893	925	956
38	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
39	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
41	922	953	983	*014	*045	*076	*106	*137	*168	*198
42	15 229	259	290	320	351	381	412	442	473	503
43	534	564	594	625	655	685	715	746	776	806
44	836	866	897	927	957	987	*017	*047	*077	*107
45	16 137	167	197	227	256	286	316	346	376	406
46	435	465	495	524	554	584	613	643	673	702
47	732	761	791	820	850	879	909	938	967	997
48	17 026	056	085	114	143	173	202	231	260	289
49	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N	0	1	2	3	4	5	6	7	8	9

P P

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

P P

150–200

N	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
51	898	926	955	984	*013	*041	*070	*099	*127	*156
52	18 184	213	241	270	298	327	355	384	412	441
53	469	498	526	554	583	611	639	667	696	724
54	752	780	808	837	865	893	921	949	977	*005
55	19 033	061	089	117	145	173	201	229	257	285
56	312	340	368	396	424	451	479	507	535	562
57	590	618	645	673	700	728	756	783	811	838
58	866	893	921	948	976	*003	*030	*058	*085	*112
59	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
61	683	710	737	763	790	817	844	871	898	925
62	952	978	*005	*032	*059	*085	*112	*139	*165	*192
63	21 219	245	272	299	325	352	378	405	431	458
64	484	511	537	564	590	617	643	669	696	722
65	748	775	801	827	854	880	906	932	958	985
66	22 011	037	063	089	115	141	167	194	220	246
67	272	298	324	350	376	401	427	453	479	505
68	531	557	583	608	634	660	686	712	737	763
69	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
71	300	325	350	376	401	426	452	477	502	528
72	553	578	603	629	654	679	704	729	754	779
73	805	830	855	880	905	930	955	980	*005	*030
74	24 055	080	105	130	155	180	204	229	254	279
75	304	329	353	378	403	428	452	477	502	527
76	551	576	601	625	650	674	699	724	748	773
77	797	822	846	871	895	920	944	969	993	*018
78	25 042	066	091	115	139	164	188	212	237	261
79	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
81	768	792	816	840	864	888	912	935	959	983
82	26 007	031	055	079	102	126	150	174	198	221
83	245	269	293	316	340	364	387	411	435	458
84	482	505	529	553	576	600	623	647	670	694
85	717	741	764	788	811	834	858	881	905	928
86	951	975	998	*021	*045	*068	*091	*114	*138	*161
87	27 184	207	231	254	277	300	323	346	370	393
88	416	439	462	485	508	531	554	577	600	623
89	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
91	28 103	126	149	171	194	217	240	262	285	307
92	330	353	375	398	421	443	466	488	511	533
93	556	578	601	623	646	668	691	713	735	758
94	780	803	825	847	870	892	914	937	959	981
95	29 003	026	048	070	092	115	137	159	181	203
96	226	248	270	292	314	336	358	380	403	425
97	447	469	491	513	535	557	579	601	623	645
98	667	688	710	732	754	776	798	820	842	863
99	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N	0	1	2	3	4	5	6	7	8	9

P P

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
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3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
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3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
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2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
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7	15.4	14.7
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9	19.8	18.9

P P

200–250

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200	30 103	125	146	168	190	211	233	255	276	298
01	320	341	363	384	406	428	449	471	492	514
02	535	557	578	600	621	643	664	685	707	728
03	750	771	792	814	835	856	878	899	920	942
04	963	984	*006	*027	*048	*069	*091	*112	*133	*154
05	31 175	197	218	239	260	281	302	323	345	366
06	387	408	429	450	471	492	513	534	555	576
07	597	618	639	660	681	702	723	744	765	785
08	806	827	848	869	890	911	931	952	973	994
09	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
11	428	449	469	490	510	531	552	572	593	613
12	634	654	675	695	715	736	756	777	797	818
13	838	858	879	899	919	940	960	980	*001	*021
14	33 041	062	082	102	122	143	163	183	203	224
15	244	264	284	304	325	345	365	385	405	425
16	445	465	486	506	526	546	566	586	606	626
17	646	666	686	706	726	746	766	786	806	826
18	846	866	885	905	925	945	965	985	*005	*025
19	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
21	439	459	479	498	518	537	557	577	596	616
22	635	655	674	694	713	733	753	772	792	811
23	830	850	869	889	908	928	947	967	986	*005
24	35 025	044	064	083	102	122	141	160	180	199
25	218	238	257	276	295	315	334	353	372	392
26	411	430	449	468	488	507	526	545	564	583
27	603	622	641	660	679	698	717	736	755	774
28	793	813	832	851	870	889	908	927	946	965
29	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
31	361	380	399	418	436	455	474	493	511	530
32	549	568	586	605	624	642	661	680	698	717
33	736	754	773	791	810	829	847	866	884	903
34	922	940	959	977	996	*014	*033	*051	*070	*088
35	37 107	125	144	162	181	199	218	236	254	273
36	291	310	328	346	365	383	401	420	438	457
37	475	493	511	530	548	566	585	603	621	639
38	658	676	694	712	731	749	767	785	803	822
39	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
41	202	220	238	256	274	292	310	328	346	364
42	382	399	417	435	453	471	489	507	525	543
43	561	578	596	614	632	650	668	686	703	721
44	739	757	775	792	810	828	846	863	881	899
45	917	934	952	970	987	*005	*023	*041	*058	*076
46	39 094	111	129	146	164	182	199	217	235	252
47	270	287	305	322	340	358	375	393	410	428
48	445	463	480	498	515	533	550	568	585	602
49	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N	0	1	2	3	4	5	6	7	8	9

PP

	22	21
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7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	20
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8	16.0
9	18.0

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5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
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3	5.4
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5	9.0
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9	16.2

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9	15.3

PP

250–300

N	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
51	967	985	*002	*019	*037	*054	*071	*088	*106	*123
52	40 140	157	175	192	209	226	243	261	278	295
53	312	329	346	364	381	398	415	432	449	466
54	483	500	518	535	552	569	586	603	620	637
55	654	671	688	705	722	739	756	773	790	807
56	824	841	858	875	892	909	926	943	960	976
57	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
58	41 162	179	196	212	229	246	263	280	296	313
59	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
61	664	681	697	714	731	747	764	780	797	814
62	830	847	863	880	896	913	929	946	963	979
63	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
64	42 160	177	193	210	226	243	259	275	292	308
65	325	341	357	374	390	406	423	439	455	472
66	488	504	521	537	553	570	586	602	619	635
67	651	667	684	700	716	732	749	765	781	797
68	813	830	846	862	878	894	911	927	943	959
69	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
71	297	313	329	345	361	377	393	409	425	441
72	457	473	489	505	521	537	553	569	584	600
73	616	632	648	664	680	696	712	727	743	759
74	775	791	807	823	838	854	870	886	902	917
75	933	949	965	981	996	*012	*028	*044	*059	*075
76	44 091	107	122	138	154	170	185	201	217	232
77	248	264	279	295	311	326	342	358	373	389
78	404	420	436	451	467	483	498	514	529	545
79	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
81	871	886	902	917	932	948	963	979	994	*010
82	45 025	040	056	071	086	102	117	133	148	163
83	179	194	209	225	240	255	271	286	301	317
84	332	347	362	378	393	408	423	439	454	469
85	484	500	515	530	545	561	576	591	606	621
86	637	652	667	682	697	712	728	743	758	773
87	788	803	818	834	849	864	879	894	909	924
88	939	954	969	984	*000	*015	*030	*045	*060	*075
89	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
91	389	404	419	434	449	464	479	494	509	523
92	538	553	568	583	598	613	627	642	657	672
93	687	702	716	731	746	761	776	790	805	820
94	835	850	864	879	894	909	923	938	953	967
95	982	997	*012	*026	*041	*056	*070	*085	*100	*114
96	47 129	144	159	173	188	202	217	232	246	261
97	276	290	305	319	334	349	363	378	392	407
98	422	436	451	465	480	494	509	524	538	553
99	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

P P

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P P

300–350

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300	47 712	727	741	756	770	784	799	813	828	842
01	857	871	885	900	914	929	943	958	972	986
02	48 001	015	029	044	058	073	087	101	116	130
03	144	159	173	187	202	216	230	244	259	273
04	287	302	316	330	344	359	373	387	401	416
05	430	444	458	473	487	501	515	530	544	558
06	572	586	601	615	629	643	657	671	686	700
07	714	728	742	756	770	785	799	813	827	841
08	855	869	883	897	911	926	940	954	968	982
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
11	276	290	304	318	332	346	360	374	388	402
12	415	429	443	457	471	485	499	513	527	541
13	554	568	582	596	610	624	638	651	665	679
14	693	707	721	734	748	762	776	790	803	817
15	831	845	859	872	886	900	914	927	941	955
16	969	982	996	*010	*024	*037	*051	*065	*079	*092
17	50 106	120	133	147	161	174	188	202	215	229
18	243	256	270	284	297	311	325	338	352	365
19	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
21	651	664	678	691	705	718	732	745	759	772
22	786	799	813	826	840	853	866	880	893	907
23	920	934	947	961	974	987	*001	*014	*028	*041
24	51 055	068	081	095	108	121	135	148	162	175
25	188	202	215	228	242	255	268	282	295	308
26	322	335	348	362	375	388	402	415	428	441
27	455	468	481	495	508	521	534	548	561	574
28	587	601	614	627	640	654	667	680	693	706
29	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
31	983	996	*009	*022	*035	*048	*061	*075	*088	*101
32	52 114	127	140	153	166	179	192	205	218	231
33	244	257	270	284	297	310	323	336	349	362
34	375	388	401	414	427	440	453	466	479	492
35	504	517	530	543	556	569	582	595	608	621
36	634	647	660	673	686	699	711	724	737	750
37	763	776	789	802	815	827	840	853	866	879
38	892	905	917	930	943	956	969	982	994	*007
39	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
41	275	288	301	314	326	339	352	364	377	390
42	403	415	428	441	453	466	479	491	504	517
43	529	542	555	567	580	593	605	618	631	643
44	656	668	681	694	706	719	732	744	757	769
45	782	794	807	820	832	845	857	870	882	895
46	908	920	933	945	958	970	983	995	*008	*020
47	54 033	045	058	070	083	095	108	120	133	145
48	158	170	183	195	208	220	233	245	258	270
49	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
N	0	1	2	3	4	5	6	7	8	9

PP

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8	11.2
9	12.6

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8	10.4
9	11.7

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7	8.4
8	9.6
9	10.8

PP

350–400

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350	54	407	419	432	444	456	469	481	494	506	518
51		531	543	555	568	580	593	605	617	630	642
52		654	667	679	691	704	716	728	741	753	765
53		777	790	802	814	827	839	851	864	876	888
54		900	913	925	937	949	962	974	986	998	*011
55	55	023	035	047	060	072	084	096	108	121	133
56		145	157	169	182	194	206	218	230	242	255
57		267	279	291	303	315	328	340	352	364	376
58		388	400	413	425	437	449	461	473	485	497
59		509	522	534	546	558	570	582	594	606	618
360		630	642	654	666	678	691	703	715	727	739
61		751	763	775	787	799	811	823	835	847	859
62		871	883	895	907	919	931	943	955	967	979
63		991	*003	*015	*027	*038	*050	*062	*074	*086	*098
64	56	110	122	134	146	158	170	182	194	205	217
65		229	241	253	265	277	289	301	312	324	336
66		348	360	372	384	396	407	419	431	443	455
67		467	478	490	502	514	526	538	549	561	573
68		585	597	608	620	632	644	656	667	679	691
69		703	714	726	738	750	761	773	785	797	808
370		820	832	844	855	867	879	891	902	914	926
71		937	949	961	972	984	996	*008	*019	*031	*043
72	57	054	066	078	089	101	113	124	136	148	159
73		171	183	194	206	217	229	241	252	264	276
74		287	299	310	322	334	345	357	368	380	392
75		403	415	426	438	449	461	473	484	496	507
76		519	530	542	553	565	576	588	600	611	623
77		634	646	657	669	680	692	703	715	726	738
78		749	761	772	784	795	807	818	830	841	852
79		864	875	887	898	910	921	933	944	955	967
380		978	990	*001	*013	*024	*035	*047	*058	*070	*081
81	58	092	104	115	127	138	149	161	172	184	195
82		206	218	229	240	252	263	274	286	297	309
83		320	331	343	354	365	377	388	399	410	422
84		433	444	456	467	478	490	501	512	524	535
85		546	557	569	580	591	602	614	625	636	647
86		659	670	681	692	704	715	726	737	749	760
87		771	782	794	805	816	827	838	850	861	872
88		883	894	906	917	928	939	950	961	973	984
89		995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59	106	118	129	140	151	162	173	184	195	207
91		218	229	240	251	262	273	284	295	306	318
92		329	340	351	362	373	384	395	406	417	428
93		439	450	461	472	483	494	506	517	528	539
94		550	561	572	583	594	605	616	627	638	649
95		660	671	682	693	704	715	726	737	748	759
96		770	780	791	802	813	824	835	846	857	868
97		879	890	901	912	923	934	945	956	966	977
98		988	999	*010	*021	*032	*043	*054	*065	*076	*086
99	60	097	108	119	130	141	152	163	173	184	195
400		206	217	228	239	249	260	271	282	293	304
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PP

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9	11.7

	12
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2	2.4
3	3.6
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7	8.4
9	9.6
9	10.8

	11
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4	4.4
5	5.5
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9	9.9

	10
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2	2.0
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7	7.0
8	8.0
9	9.0

PP

400–450

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400	60 206	217	228	239	249	260	271	282	293	304
01	314	325	336	347	358	369	379	390	401	412
02	423	433	444	455	466	477	487	498	509	520
03	531	541	552	563	574	584	595	606	617	627
04	638	649	660	670	681	692	703	713	724	735
05	746	756	767	778	788	799	810	821	831	842
06	853	863	874	885	895	906	917	927	938	949
07	959	970	981	991	*002	*013	*023	*034	*045	*055
08	61 066	077	087	098	109	119	130	140	151	162
09	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
11	384	395	405	416	426	437	448	458	469	479
12	490	500	511	521	532	542	553	563	574	584
13	595	606	616	627	637	648	658	669	679	690
14	700	711	721	731	742	752	763	773	784	794
15	805	815	826	836	847	857	868	878	888	899
16	909	920	930	941	951	962	972	982	993	*003
17	62 014	024	034	045	055	066	076	086	097	107
18	118	128	138	149	159	170	180	190	201	211
19	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
21	428	439	449	459	469	480	490	500	511	521
22	531	542	552	562	572	583	593	603	613	624
23	634	644	655	665	675	685	696	706	716	726
24	737	747	757	767	778	788	798	808	818	829
25	839	849	859	870	880	890	900	910	921	931
26	941	951	961	972	982	992	*002	*012	*022	*033
27	63 043	053	063	073	083	094	104	114	124	134
28	144	155	165	175	185	195	205	215	225	236
29	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
31	448	458	468	478	488	498	508	518	528	538
32	548	558	568	579	589	599	609	619	629	639
33	649	659	669	679	689	699	709	719	729	739
34	749	759	769	779	789	799	809	819	829	839
35	849	859	869	879	889	899	909	919	929	939
36	949	959	969	979	988	998	*008	*018	*028	*038
37	64 048	058	068	078	088	098	108	118	128	137
38	147	157	167	177	187	197	207	217	227	237
39	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
41	444	454	464	473	483	493	503	513	523	532
42	542	552	562	572	582	591	601	611	621	631
43	640	650	660	670	680	689	699	709	719	729
44	738	748	758	768	777	787	797	807	816	826
45	836	846	856	865	875	885	895	904	914	924
46	933	943	953	963	972	982	992	*002	*011	*021
47	65 031	040	050	060	070	079	089	099	108	118
48	128	137	147	157	167	176	186	196	205	215
49	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408
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PP

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PP

450–500

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450	65 321	331	341	350	360	369	379	389	398	408
51	418	427	437	447	456	466	475	485	495	504
52	514	523	533	543	552	562	571	581	591	600
53	610	619	629	639	648	658	667	677	686	696
54	706	715	725	734	744	753	763	772	782	792
55	801	811	820	830	839	849	858	868	877	887
56	896	906	916	925	935	944	954	963	973	982
57	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
58	66 087	096	106	115	124	134	143	153	162	172
59	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
61	370	380	389	398	408	417	427	436	445	455
62	464	474	483	492	502	511	521	530	539	549
63	558	567	577	586	596	605	614	624	633	642
64	652	661	671	680	689	699	708	717	727	736
65	745	755	764	773	783	792	801	811	820	829
66	839	848	857	867	876	885	894	904	913	922
67	932	941	950	960	969	978	987	997	*006	*015
68	67 025	034	043	052	062	071	080	089	099	108
69	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
71	302	311	321	330	339	348	357	367	376	385
72	394	403	413	422	431	440	449	459	468	477
73	486	495	504	514	523	532	541	550	560	569
74	578	587	596	605	614	624	633	642	651	660
75	669	679	688	697	706	715	724	733	742	752
76	761	770	779	788	797	806	815	825	834	843
77	852	861	870	879	888	897	906	916	925	934
78	943	952	961	970	979	988	997	*006	*015	*024
79	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
81	215	224	233	242	251	260	269	278	287	296
82	305	314	323	332	341	350	359	368	377	386
83	395	404	413	422	431	440	449	458	467	476
84	485	494	502	511	520	529	538	547	556	565
85	574	583	592	601	610	619	628	637	646	655
86	664	673	681	690	699	708	717	726	735	744
87	753	762	771	780	789	797	806	815	824	833
88	842	851	860	869	878	886	895	904	913	922
89	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
91	108	117	126	135	144	152	161	170	179	188
92	197	205	214	223	232	241	249	258	267	276
93	285	294	302	311	320	329	338	346	355	364
94	373	381	390	399	408	417	425	434	443	452
95	461	469	478	487	496	504	513	522	531	539
96	548	557	566	574	583	592	601	609	618	627
97	636	644	653	662	671	679	688	697	705	714
98	723	732	740	749	758	767	775	784	793	801
99	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975
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PP

500–550

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500	69 897	906	914	923	932	940	949	958	966	975	
01	984	992	*001	*010	*018	*027	*036	*044	*053	*062	
02	70 070	079	088	096	105	114	122	131	140	148	
03	157	165	174	183	191	200	209	217	226	234	
04	243	252	260	269	278	286	295	303	312	321	
05	329	338	346	355	364	372	381	389	398	406	
06	415	424	432	441	449	458	467	475	484	492	9 1 0.9 2 1.8 3 2.7 4 3.6 5 4.5 6 5.4 7 6.3 8 7.2 9 8.1
07	501	509	518	526	535	544	552	561	569	578	
08	586	595	603	612	621	629	638	646	655	663	
09	672	680	689	697	706	714	723	731	740	749	
510	757	766	774	783	791	800	808	817	825	834	
11	842	851	859	868	876	885	893	902	910	919	
12	927	935	944	952	961	969	978	986	995	*003	
13	71 012	020	029	037	046	054	063	071	079	088	
14	096	105	113	122	130	139	147	155	164	172	
15	181	189	198	206	214	223	231	240	248	257	
16	265	273	282	290	299	307	315	324	332	341	
17	349	357	366	374	383	391	399	408	416	425	
18	433	441	450	458	466	475	483	492	500	508	
19	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	
21	684	692	700	709	717	725	734	742	750	759	8 1 0.8 2 1.6 3 2.4 4 3.2 5 4.0 6 4.8 7 5.6 8 6.4 9 7.2
22	767	775	784	792	800	809	817	825	834	842	
23	850	858	867	875	883	892	900	908	917	925	
24	933	941	950	958	966	975	983	991	999	*008	
25	72 016	024	032	041	049	057	066	074	082	090	
26	099	107	115	123	132	140	148	156	165	173	
27	181	189	198	206	214	222	230	239	247	255	
28	263	272	280	288	296	304	313	321	329	337	
29	346	354	362	370	378	387	395	403	411	419	
530	428	436	444	452	460	469	477	485	493	501	
31	509	518	526	534	542	550	558	567	575	583	
32	591	599	607	616	624	632	640	648	656	665	
33	673	681	689	697	705	713	722	730	738	746	
34	754	762	770	779	787	795	803	811	819	827	
35	835	843	852	860	868	876	884	892	900	908	
36	916	925	933	941	949	957	965	973	981	989	7 1 0.7 2 1.4 3 2.1 4 2.8 5 3.5 6 4.2 7 4.9 8 5.6 9 6.3
37	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	
38	73 078	086	094	102	111	119	127	135	143	151	
39	159	167	175	183	191	199	207	215	223	231	
540	239	247	255	263	272	280	288	296	304	312	
41	320	328	336	344	352	360	368	376	384	392	
42	400	408	416	424	432	440	448	456	464	472	
43	480	488	496	504	512	520	528	536	544	552	
44	560	568	576	584	592	600	608	616	624	632	
45	640	648	656	664	672	679	687	695	703	711	
46	719	727	735	743	751	759	767	775	783	791	
47	799	807	815	823	830	838	846	854	862	870	
48	878	886	894	902	910	918	926	933	941	949	
49	957	965	973	981	989	997	*005	*013	*020	*028	
550	74 036	044	052	060	068	076	084	092	099	107	
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550–600

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550	74 036	044	052	060	068	076	084	092	099	107
51	115	123	131	139	147	155	162	170	178	186
52	194	202	210	218	225	233	241	249	257	265
53	273	280	288	296	304	312	320	327	335	343
54	351	359	367	374	382	390	398	406	414	421
55	429	437	445	453	461	468	476	484	492	500
56	507	515	523	531	539	547	554	562	570	578
57	586	593	601	609	617	624	632	640	648	656
58	663	671	679	687	695	702	710	718	726	733
59	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
61	896	904	912	920	927	935	943	950	958	966
62	974	981	989	997	*005	*012	*020	*028	*035	*043
63	75 051	059	066	074	082	089	097	105	113	120
64	128	136	143	151	159	166	174	182	189	197
65	205	213	220	228	236	243	251	259	266	274
66	282	289	297	305	312	320	328	335	343	351
67	358	366	374	381	389	397	404	412	420	427
68	435	442	450	458	465	473	481	488	496	504
69	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
71	664	671	679	686	694	702	709	717	724	732
72	740	747	755	762	770	778	785	793	800	808
73	815	823	831	838	846	853	861	868	876	884
74	891	899	906	914	921	929	937	944	952	959
75	967	974	982	989	997	*005	*012	*020	*027	*035
76	76 042	050	057	065	072	080	087	095	103	110
77	118	125	133	140	148	155	163	170	178	185
78	193	200	208	215	223	230	238	245	253	260
79	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
81	418	425	433	440	448	455	462	470	477	485
82	492	500	507	515	522	530	537	545	552	559
83	567	574	582	589	597	604	612	619	626	634
84	641	649	656	664	671	678	686	693	701	708
85	716	723	730	738	745	753	760	768	775	782
86	790	797	805	812	819	827	834	842	849	856
87	864	871	879	886	893	901	908	916	923	930
88	938	945	953	960	967	975	982	989	997	*004
89	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
91	159	166	173	181	188	195	203	210	217	225
92	232	240	247	254	262	269	276	283	291	298
93	305	313	320	327	335	342	349	357	364	371
94	379	386	393	401	408	415	422	430	437	444
95	452	459	466	474	481	488	495	503	510	517
96	525	532	539	546	554	561	568	576	583	590
97	597	605	612	619	627	634	641	648	656	663
98	670	677	685	692	699	706	714	721	728	735
99	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
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PP

600–650

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600	77 815	822	830	837	844	851	859	866	873	880	
01	887	895	902	909	916	924	931	938	945	952	
02	960	967	974	981	988	996	*003	*010	*017	*025	
03	78 032	039	046	053	061	068	075	082	089	097	
04	104	111	118	125	132	140	147	154	161	168	
05	176	183	190	197	204	211	219	226	233	240	
06	247	254	262	269	276	283	290	297	305	312	
07	319	326	333	340	347	355	362	369	376	383	
08	390	398	405	412	419	426	433	440	447	455	
09	462	469	476	483	490	497	504	512	519	526	
610	533	540	547	554	561	569	576	583	590	597	
11	604	611	618	625	633	640	647	654	661	668	
12	675	682	689	696	704	711	718	725	732	739	
13	746	753	760	767	774	781	789	796	803	810	
14	817	824	831	838	845	852	859	866	873	880	
15	888	895	902	909	916	923	930	937	944	951	
16	958	965	972	979	986	993	*000	*007	*014	*021	
17	79 029	036	043	050	057	064	071	078	085	092	
18	099	106	113	120	127	134	141	148	155	162	
19	169	176	183	190	197	204	211	218	225	232	
620	239	246	253	260	267	274	281	288	295	302	
21	309	316	323	330	337	344	351	358	365	372	
22	379	386	393	400	407	414	421	428	435	442	
23	449	456	463	470	477	484	491	498	505	511	
24	518	525	532	539	546	553	560	567	574	581	
25	588	595	602	609	616	623	630	637	644	650	
26	657	664	671	678	685	692	699	706	713	720	
27	727	734	741	748	754	761	768	775	782	789	
28	796	803	810	817	824	831	837	844	851	858	
29	865	872	879	886	893	900	906	913	920	927	
630	934	941	948	955	962	969	975	982	989	996	
31	80 003	010	017	024	030	037	044	051	058	065	
32	072	079	085	092	099	106	113	120	127	134	
33	140	147	154	161	168	175	182	188	195	202	
34	209	216	223	229	236	243	250	257	264	271	
35	277	284	291	298	305	312	318	325	332	339	
36	346	353	359	366	373	380	387	393	400	407	
37	414	421	428	434	441	448	455	462	468	475	
38	482	489	496	502	509	516	523	530	536	543	
39	550	557	564	570	577	584	591	598	604	611	
640	618	625	632	638	645	652	659	665	672	679	
41	686	693	699	706	713	720	726	733	740	747	
42	754	760	767	774	781	787	794	801	808	814	
43	821	828	835	841	848	855	862	868	875	882	
44	889	895	902	909	916	922	929	936	943	949	
45	956	963	969	976	983	990	996	*003	*010	*017	
46	81 023	030	037	043	050	057	064	070	077	084	
47	090	097	104	111	117	124	131	137	144	151	
48	158	164	171	178	184	191	198	204	211	218	
49	224	231	238	245	251	258	265	271	278	285	
650	291	298	305	311	318	325	331	338	345	351	
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650–700

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650	81 291	298	305	311	318	325	331	338	345	351	
51	358	365	371	378	385	391	398	405	411	418	
52	425	431	438	445	451	458	465	471	478	485	
53	491	498	505	511	518	525	531	538	544	551	
54	558	564	571	578	584	591	598	604	611	617	
55	624	631	637	644	651	657	664	671	677	684	
56	690	697	704	710	717	723	730	737	743	750	
57	757	763	770	776	783	790	796	803	809	816	
58	823	829	836	842	849	856	862	869	875	882	
59	889	895	902	908	915	921	928	935	941	948	
660	954	961	968	974	981	987	994	*000	*007	*014	
61	82 020	027	033	040	046	053	060	066	073	079	
62	086	092	099	105	112	119	125	132	138	145	
63	151	158	164	171	178	184	191	197	204	210	
64	217	223	230	236	243	249	256	263	269	276	
65	282	289	295	302	308	315	321	328	334	341	
66	347	354	360	367	373	380	387	393	400	406	
67	413	419	426	432	439	445	452	458	465	471	
68	478	484	491	497	504	510	517	523	530	536	
69	543	549	556	562	569	575	582	588	595	601	
670	607	614	620	627	633	640	646	653	659	666	
71	672	679	685	692	698	705	711	718	724	730	
72	737	743	750	756	763	769	776	782	789	795	
73	802	808	814	821	827	834	840	847	853	860	
74	866	872	879	885	892	898	905	911	918	924	
75	930	937	943	950	956	963	969	975	982	988	
76	995	*001	*008	*014	*020	*027	*033	*040	*046	*052	
77	83 059	065	072	078	085	091	097	104	110	117	
78	123	129	136	142	149	155	161	168	174	181	
79	187	193	200	206	213	219	225	232	238	245	
680	251	257	264	270	276	283	289	296	302	308	
81	315	321	327	334	340	347	353	359	366	372	
82	378	385	391	398	404	410	417	423	429	436	
83	442	448	455	461	467	474	480	487	493	499	
84	506	512	518	525	531	537	544	550	556	563	
85	569	575	582	588	594	601	607	613	620	626	
86	632	639	645	651	658	664	670	677	683	689	
87	696	702	708	715	721	727	734	740	746	753	
88	759	765	771	778	784	790	797	803	809	816	
89	822	828	835	841	847	853	860	866	872	879	
690	885	891	897	904	910	916	923	929	935	942	
91	948	954	960	967	973	979	985	992	998	*004	
92	84 011	017	023	029	036	042	048	055	061	067	
93	073	080	086	092	098	105	111	117	123	130	
94	136	142	148	155	161	167	173	180	186	192	
95	198	205	211	217	223	230	236	242	248	255	
96	261	267	273	280	286	292	298	305	311	317	
97	323	330	336	342	348	354	361	367	373	379	
98	386	392	398	404	410	417	423	429	435	442	
99	448	454	460	466	473	479	485	491	497	504	
700	510	516	522	528	535	541	547	553	559	566	
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9	6.3

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8	4.8
9	5.4

700–750

N	0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
01	572	578	584	590	597	603	609	615	621	628
02	634	640	646	652	658	665	671	677	683	689
03	696	702	708	714	720	726	733	739	745	751
04	757	763	770	776	782	788	794	800	807	813
05	819	825	831	837	844	850	856	862	868	874
06	880	887	893	899	905	911	917	924	930	936
07	942	948	954	960	967	973	979	985	991	997
08	85 003	009	016	022	028	034	040	046	052	058
09	065	071	077	083	089	095	101	107	114	120
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19	673	679	685	691	697	703	709	715	721	727
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21	794	800	806	812	818	824	830	836	842	848
22	854	860	866	872	878	884	890	896	902	908
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24	974	980	986	992	998	*004	*010	*016	*022	*028
25	86 034	040	046	052	058	064	070	076	082	088
26	094	100	106	112	118	124	130	136	141	147
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35	629	635	641	646	652	658	664	670	676	682
36	688	694	700	705	711	717	723	729	735	741
37	747	753	759	764	770	776	782	788	794	800
38	806	812	817	823	829	835	841	847	853	859
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41	982	988	994	999	*005	*011	*017	*023	*029	*035
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46	274	280	286	291	297	303	309	315	320	326
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48	390	396	402	408	413	419	425	431	437	442
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52	622	628	633	639	645	651	656	662	668	574	
53	679	685	691	697	703	708	714	720	726	731	
54	737	743	749	754	760	766	772	777	783	789	
55	795	800	806	812	818	823	829	835	841	846	
56	852	858	864	869	875	881	887	892	898	904	
57	910	915	921	927	933	938	944	950	955	961	
58	967	973	978	984	990	996	*001	*007	*013	*018	
59	88 024	030	036	041	047	053	058	064	070	076	
760	081	087	093	098	104	110	116	121	127	133	
61	138	144	150	156	161	167	173	178	184	190	
62	195	201	207	213	218	224	230	235	241	247	
63	252	258	264	270	275	281	287	292	298	304	
64	309	315	321	326	332	338	343	349	355	360	
65	366	372	377	383	389	395	400	406	412	417	
66	423	429	434	440	446	451	457	463	468	474	
67	480	485	491	497	502	508	513	519	525	530	
68	536	542	547	553	559	564	570	576	581	587	
69	593	598	604	610	615	621	627	632	638	643	
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71	705	711	717	722	728	734	739	745	750	756	
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82	321	326	332	337	343	348	354	360	365	371	
83	376	382	387	393	398	404	409	415	421	426	
84	432	437	443	448	454	459	465	470	476	481	
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87	597	603	609	614	620	625	631	636	642	647	
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92	873	878	883	889	894	900	905	911	916	922	
93	927	933	938	944	949	955	960	966	971	977	
94	982	988	993	998	*004	*009	*015	*020	*026	*031	
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97	146	151	157	162	168	173	179	184	189	195	
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07	687	693	698	703	709	714	720	725	730	736
08	741	747	752	757	763	768	773	779	784	789
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16	169	174	180	185	190	196	201	206	212	217
17	222	228	233	238	243	249	254	259	265	270
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23	540	545	551	556	561	566	572	577	582	587
24	593	598	603	609	614	619	624	630	635	640
25	645	651	656	661	666	672	677	682	687	693
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31	960	965	971	976	981	986	991	997	*002	*007
32	92 012	018	023	028	033	038	044	049	054	059
33	065	070	075	080	085	091	096	101	106	111
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36	221	226	231	236	241	247	252	257	262	267
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44	634	639	645	650	655	660	665	670	675	681
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46	737	742	747	752	758	763	768	773	778	783
47	788	793	799	804	809	814	819	824	829	834
48	840	845	850	855	860	865	870	875	881	886
49	891	896	901	906	911	916	921	927	932	937
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55	197	202	207	212	217	222	227	232	237	242	
56	247	252	258	263	268	273	278	283	288	293	
57	298	303	308	313	318	323	328	334	339	344	
58	349	354	359	364	369	374	379	384	389	394	
59	399	404	409	414	420	425	430	435	440	445	
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61	500	505	510	515	520	526	531	536	541	546	
62	551	556	561	566	571	576	581	586	591	596	
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71	94 002	007	012	017	022	027	032	037	042	047	
72	052	057	062	067	072	077	082	086	091	096	
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82	547	552	557	562	567	571	576	581	586	591	
83	596	601	606	611	616	621	626	630	635	640	
84	645	650	655	660	665	670	675	680	685	689	
85	694	699	704	709	714	719	724	729	734	738	
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91	988	993	998	*002	*007	*012	*017	*022	*027	*032	
92	95 036	041	046	051	056	061	066	071	075	080	
93	085	090	095	100	105	109	114	119	124	129	
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900–950

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02	521	525	530	535	540	545	550	554	559	564
03	569	574	578	583	588	593	598	602	607	612
04	617	622	626	631	636	641	646	650	655	660
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25	614	619	624	628	633	638	642	647	652	656
26	661	666	670	675	680	685	689	694	699	703
27	708	713	717	722	727	731	736	741	745	750
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33	988	993	997	*002	*007	*011	*016	*021	*025	*030
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71	722	726	731	735	740	744	749	753	758	762	
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73	811	816	820	825	829	834	838	843	847	851	
74	856	860	865	869	874	878	883	887	892	896	
75	900	905	909	914	918	923	927	932	936	941	
76	945	949	954	958	963	967	972	976	981	985	
77	989	994	998	*003	*007	*012	*016	*021	*025	*029	
78	99 034	038	043	047	052	056	061	065	069	074	
79	078	083	087	092	096	100	105	109	114	118	
980	123	127	131	136	140	145	149	154	158	162	
81	167	171	176	180	185	189	193	198	202	207	
82	211	216	220	224	229	233	238	242	247	251	
83	255	260	264	269	273	277	282	286	291	295	
84	300	304	308	313	317	322	326	330	335	339	
85	344	348	352	357	361	366	370	374	379	383	
86	388	392	396	401	405	410	414	419	423	427	
87	432	436	441	445	449	454	458	463	467	471	
88	476	480	484	489	493	498	502	506	511	515	
89	520	524	528	533	537	542	546	550	555	559	
990	564	568	572	577	581	585	590	594	599	603	
91	607	612	616	621	625	629	634	638	642	647	
92	651	656	660	664	669	673	677	682	686	691	
93	695	699	704	708	712	717	721	726	730	734	
94	739	743	747	752	756	760	765	769	774	778	
95	782	787	791	795	800	804	808	813	817	822	
96	826	830	835	839	843	848	852	856	861	865	
97	870	874	878	883	887	891	896	900	904	909	
98	913	917	922	926	930	935	939	944	948	952	
99	957	961	965	970	974	978	983	987	991	996	
1000	00 000	004	009	013	017	022	026	030	035	039	
N	0	1	2	3	4	5	6	7	8	9	PP

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Table II

POWERS AND ROOTS

No.	Square	Cube	Sq Root	Cu Root
1	1	1	1.0000	1.0000
2	4	8	1.4142	1.2599
3	9	27	1.7321	1.4422
4	16	64	2.0000	1.5874
5	25	125	2.2361	1.7100
6	36	216	2.4495	1.8171
7	49	343	2.6458	1.9129
8	64	512	2.8284	2.0000
9	81	729	3.0000	2.0801
10	100	1,000	3.1623	2.1544
11	121	1,331	3.3166	2.2240
12	144	1,728	3.4641	2.2894
13	169	2,197	3.6056	2.3513
14	196	2,744	3.7417	2.4101
15	225	3,375	3.8730	2.4662
16	256	4,096	4.0000	2.5198
17	289	4,913	4.1231	2.5713
18	324	5,832	4.2426	2.6207
19	361	6,859	4.3589	2.6684
20	400	8,000	4.4721	2.7144
21	441	9,261	4.5826	2.7589
22	484	10,648	4.6904	2.8020
23	529	12,167	4.7958	2.8439
24	576	13,824	4.8990	2.8845
25	625	15,625	5.0000	2.9240
26	676	17,576	5.0990	2.9625
27	729	19,683	5.1962	3.0000
28	784	21,952	5.2915	3.0366
29	841	24,389	5.3852	3.0723
30	900	27,000	5.4772	3.1072
31	961	29,791	5.5678	3.1414
32	1,024	32,768	5.6569	3.1748
33	1,089	35,937	5.7446	3.2075
34	1,156	39,304	5.8310	3.2396
35	1,225	42,875	5.9161	3.2711
36	1,296	46,656	6.0000	3.3019
37	1,369	50,653	6.0828	3.3322
38	1,444	54,872	6.1644	3.3620
39	1,521	59,319	6.2450	3.3912
40	1,600	64,000	6.3246	3.4200
41	1,681	68,921	6.4031	3.4482
42	1,764	74,088	6.4807	3.4760
43	1,849	79,507	6.5574	3.5034
44	1,936	85,184	6.6332	3.5303
45	2,025	91,125	6.7082	3.5569
46	2,116	97,336	6.7823	3.5830
47	2,209	103,823	6.8557	3.6088
48	2,304	110,592	6.9282	3.6342
49	2,401	117,649	7.0000	3.6593
50	2,500	125,000	7.0711	3.6840

No.	Square	Cube	Sq Root	Cu Root
50	2,500	125,000	7.0711	3.6840
51	2,601	132,651	7.1414	3.7084
52	2,704	140,608	7.2111	3.7325
53	2,809	148,877	7.2801	3.7563
54	2,916	157,464	7.3485	3.7798
55	3,025	166,375	7.4162	3.8030
56	3,136	175,616	7.4833	3.8259
57	3,249	185,193	7.5498	3.8485
58	3,364	195,112	7.6158	3.8709
59	3,481	205,379	7.6811	3.8930
60	3,600	216,000	7.7460	3.9149
61	3,721	226,981	7.8102	3.9365
62	3,844	238,328	7.8740	3.9579
63	3,969	250,047	7.9373	3.9791
64	4,096	262,144	8.0000	4.0000
65	4,225	274,625	8.0623	4.0207
66	4,356	287,496	8.1240	4.0412
67	4,489	300,763	8.1854	4.0615
68	4,624	314,432	8.2462	4.0817
69	4,761	328,509	8.3066	4.1016
70	4,900	343,000	8.3666	4.1213
71	5,041	357,911	8.4261	4.1408
72	5,184	373,248	8.4853	4.1602
73	5,329	389,017	8.5440	4.1793
74	5,476	405,224	8.6023	4.1983
75	5,625	421,875	8.6603	4.2172
76	5,776	438,976	8.7178	4.2358
77	5,929	456,533	8.7750	4.2543
78	6,084	474,552	8.8318	4.2727
79	6,241	493,039	8.8882	4.2908
80	6,400	512,000	8.9443	4.3089
81	6,561	531,441	9.0000	4.3267
82	6,724	551,368	9.0554	4.3445
83	6,889	571,787	9.1104	4.3621
84	7,056	592,704	9.1652	4.3795
85	7,225	614,125	9.2195	4.3968
86	7,396	636,056	9.2736	4.4140
87	7,569	658,503	9.3274	4.4310
88	7,744	681,472	9.3808	4.4480
89	7,921	704,969	9.4340	4.4647
90	8,100	729,000	9.4868	4.4814
91	8,281	753,571	9.5394	4.4979
92	8,464	778,688	9,5917	4.5144
93	8,649	804,357	9.6437	4.5307
94	8,836	830,584	9.6954	4.5468
95	9,025	857,375	9.7468	4.5629
96	9,216	884,736	9.7980	4.5789
97	9,409	912,673	9.8489	4.5947
98	9,604	941,192	9.8995	4.6104
99	9,801	970,299	9.9499	4.6261
100	10,000	1,000,000	10.0000	4.6416

Table III

COMPOUND AMOUNT: $(1 + r)^n$

Note: The table gives six-place values of the compound amount of 1 for values of n from 1 to 100, and certain values of r from $\frac{5}{12}\%$ to 7%. In the formula, n is the number of conversion periods and r is the rate per period. The compound amount of P is found by multiplying the tabular value by P. To find the compound amount of $120.50 for 2 years at 2% compounded quarterly, for example, we take $n = 8$ (2 times 4 periods per year) and $r = \frac{1}{2}\%$ (2% divided by 4). In the $\frac{1}{2}\%$ column opposite n equal to 8, we read 1.040707. The compound amount for $120.50 for the stated time and rate is, therefore, 120.50 × 1.040707, or $125.41.

$(1 + r)^n$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.004 167	1.005 000	1.005 833	1.007 500	1.010 000
2	1.008 351	1.010 025	1.011 701	1.015 056	1.020 100
3	1.012 552	1.015 075	1.017 602	1.022 669	1.030 301
4	1.016 771	1.020 151	1.023 538	1.030 339	1.040 604
5	1.021 008	1.025 251	1.029 509	1.038 067	1.051 010
6	1.025 262	1.030 378	1.035 514	1.045 852	1.061 520
7	1.029 534	1.035 529	1.041 555	1.053 696	1.072 135
8	1.033 824	1.040 707	1.047 631	1.061 599	1.082 857
9	1.038 131	1.045 911	1.053 742	1.069 561	1.093 685
10	1.042 457	1.051 140	1.059 889	1.077 583	1.104 622
11	1.046 800	1.056 396	1.066 071	1.085 664	1.115 668
12	1.051 162	1.061 678	1.072 290	1.093 807	1.126 825
13	1.055 542	1.066 986	1.078 545	1.102 010	1.138 093
14	1.059 940	1.072 321	1.084 837	1.110 276	1.149 474
15	1.064 356	1.077 683	1.091 165	1.118 603	1.160 969
16	1.068 791	1.083 071	1.097 530	1.126 992	1.172 579
17	1.073 244	1.088 487	1.103 932	1.135 445	1.184 304
18	1.077 716	1.093 929	1.110 372	1.143 960	1.196 147
19	1.082 207	1.099 399	1.116 849	1.152 540	1.208 109
20	1.086 716	1.104 896	1.123 364	1.161 184	1.220 190
21	1.091 244	1.110 420	1.129 917	1.169 893	1.232 392
22	1.095 791	1.115 972	1.136 508	1.178 667	1.244 716
23	1.100 357	1.121 552	1.143 138	1.187 507	1.257 163
24	1.104 941	1.127 160	1.149 806	1.196 414	1.269 735
25	1.109 545	1.132 796	1.156 513	1.205 387	1.282 432
26	1.114 168	1.138 460	1.163 260	1.214 427	1.295 256
27	1.118 811	1.144 152	1.170 045	1.223 535	1.308 209
28	1.123 472	1.149 873	1.176 870	1.232 712	1.321 291
29	1.128 154	1.155 622	1.183 736	1.241 957	1.334 504
30	1.132 854	1.161 400	1.190 641	1.251 272	1.347 849
31	1.137 574	1.167 207	1.197 586	1.260 656	1.361 327
32	1.142 314	1.173 043	1.204 572	1.270 111	1.374 941
33	1.147 074	1.178 908	1.211 599	1.279 637	1.388 690
34	1.151 853	1.184 803	1.218 666	1.289 234	1.402 577
35	1.156 653	1.190 727	1.225 775	1.298 904	1.416 603
36	1.161 472	1.196 681	1.232 926	1.308 645	1.430 769
37	1.166 312	1.202 664	1.240 118	1.318 460	1.445 076
38	1.171 171	1.208 677	1.247 352	1.328 349	1.459 527
39	1.176 051	1.214 721	1.254 628	1.338 311	1.474 123
40	1.180 951	1.220 794	1.261 947	1.348 349	1.488 864
41	1.185 872	1.226 898	1.269 308	1.358 461	1.503 752
42	1.190 813	1.233 033	1.276 712	1.368 650	1.518 790
43	1.195 775	1.239 198	1.284 160	1.378 915	1.533 978
44	1.200 757	1.245 394	1.291 651	1.389 256	1.549 318
45	1.205 760	1.251 621	1.299 185	1.399 676	1.564 811
46	1.210 784	1.257 879	1.306 764	1.410 173	1.580 459
47	1.215 829	1.264 168	1.314 387	1.420 750	1.596 263
48	1.220 895	1.270 489	1.322 054	1.431 405	1.612 226
49	1.225 982	1.276 842	1.329 766	1.442 141	1.628 348
50	1.231 091	1.283 226	1.337 523	1.452 957	1.644 632

$(1 + r)^n$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	1.236 220	1.289 642	1.345 325	1.463 854	1.661 078
52	1.241 371	1.296 090	1.353 173	1.474 833	1.677 689
53	1.246 541	1.302 571	1.361 066	1.485 894	1.694 466
54	1.251 737	1.309 083	1.369 006	1.497 038	1.711 410
55	1.256 953	1.315 629	1.376 992	1.508 266	1.728 525
56	1.262 190	1.322 207	1.385 024	1.519 578	1.745 810
57	1.267 449	1.328 818	1.393 103	1.530 975	1.763 268
58	1.272 730	1.335 462	1.401 230	1.542 457	1.780 901
59	1.278 034	1.342 139	1.409 404	1.554 026	1.798 710
60	1.283 359	1.348 850	1.417 625	1.565 681	1.816 697
61	1.288 706	1.355 594	1.425 895	1.577 424	1.834 864
62	1.294 076	1.362 372	1.434 212	1.589 254	1.853 212
63	1.299 468	1.369 184	1.442 579	1.601 174	1.871 744
64	1.304 882	1.376 030	1.450 994	1.613 183	1.890 462
65	1.310 319	1.382 910	1.459 458	1.625 281	1.909 366
66	1.315 779	1.389 825	1.467 971	1.637 471	1.928 460
67	1.321 261	1.396 774	1.476 535	1.649 752	1.947 745
68	1.326 766	1.403 758	1.485 148	1.662 125	1.967 222
69	1.332 295	1.410 777	1.493 811	1.674 591	1.986 894
70	1.337 846	1.417 831	1.502 525	1.687 151	2.006 763
71	1.343 420	1.424 920	1.511 290	1.699 804	2.026 831
72	1.349 018	1.432 044	1.520 106	1.712 553	2.047 099
73	1.354 639	1.439 204	1.528 973	1.725 397	2.067 570
74	1.360 283	1.446 401	1.537 892	1.738 337	2.088 246
75	1.365 951	1.453 633	1.546 863	1.751 375	2.109 128
76	1.371 642	1.460 901	1.555 886	1.764 510	2.130 220
77	1.377 357	1.468 205	1.564 962	1.777 744	2.151 522
78	1.383 096	1.475 546	1.574 091	1.791 077	2.173 037
79	1.388 859	1.482 924	1.583 273	1.804 510	2.194 768
80	1.394 646	1.490 339	1.592 509	1.818 044	2.216 715
81	1.400 457	1.497 790	1.601 799	1.831 679	2.238 882
82	1.406 293	1.505 279	1.611 143	1.845 417	2.261 271
83	1.412 152	1.512 806	1.620 541	1.859 258	2.283 884
84	1.418 036	1.520 370	1.629 994	1.873 202	2.306 723
85	1.423 945	1.527 971	1.639 502	1.887 251	2.329 790
86	1.429 878	1.535 611	1.649 066	1.901 405	2.353 088
87	1.435 835	1.543 289	1.658 686	1.915 666	2.376 619
88	1.441 818	1.551 006	1.668 361	1.930 033	2.400 385
89	1.447 826	1.558 761	1.678 093	1.944 509	2.424 389
90	1.453 858	1.566 555	1.687 882	1.959 092	2.448 633
91	1.459 916	1.574 387	1.697 728	1.973 786	2.473 119
92	1.465 999	1.582 259	1.707 632	1.988 589	2.497 850
93	1.472 107	1.590 171	1.717 593	2.003 503	2.522 829
94	1.478 241	1.598 122	1.727 612	2.018 530	2.548 057
95	1.484 400	1.606 112	1.737 690	2.033 669	2.573 538
96	1.490 585	1.614 143	1.747 826	2.048 921	2.599 273
97	1.496 796	1.622 213	1.758 022	2.064 288	2.625 266
98	1.503 033	1.630 324	1.768 277	2.079 770	2.651 518
99	1.509 296	1.638 776	1.778 592	2.095 369	2.678 033
100	1.515 584	1.646 668	1.788 967	2.111 084	2.704 814

$(1 + r)^n$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.011 250	1.012 500	1.015 000	1.017 500	1.020 000
2	1.022 627	1.025 156	1.030 225	1.035 306	1.040 400
3	1.034 131	1.037 971	1.045 678	1.053 424	1.061 208
4	1.045 765	1.050 945	1.061 364	1.071 859	1.082 432
5	1.057 530	1.064 082	1.077 284	1.090 617	1.104 081
6	1.069 427	1.077 383	1.093 443	1.109 702	1.126 162
7	1.081 458	1.090 850	1.109 845	1.129 122	1.148 686
8	1.093 625	1.104 486	1.126 493	1.148 882	1.171 659
9	1.105 928	1.118 292	1.143 390	1.168 987	1.195 093
10	1.118 370	1.132 271	1.160 541	1.189 444	1.218 994
11	1.130 951	1.146 424	1.177 949	1.210 260	1.243 374
12	1.143 674	1.160 755	1.195 618	1.231 439	1.268 242
13	1.156 541	1.175 264	1.213 552	1.252 990	1.293 607
14	1.169 552	1.189 955	1.231 756	1.274 917	1.319 479
15	1.182 709	1.204 829	1.250 232	1.297 228	1.345 868
16	1.196 015	1.219 890	1.268 986	1.319 929	1.372 786
17	1.209 470	1.235 138	1.288 020	1.343 028	1.400 241
18	1.223 077	1.250 577	1.307 341	1.366 531	1.428 246
19	1.236 836	1.266 210	1.326 951	1.390 445	1.456 811
20	1.250 751	1.282 037	1.346 855	1.414 778	1.485 947
21	1.264 821	1.298 063	1.367 058	1.439 537	1.515 666
22	1.279 051	1.314 288	1.387 564	1.464 729	1.545 980
23	1.293 440	1.330 717	1.408 377	1.490 361	1.576 899
24	1.307 991	1.347 351	1.429 503	1.516 443	1.608 437
25	1.322 706	1.364 193	1.450 945	1.542 981	1.640 606
26	1.337 587	1.381 245	1.472 710	1.569 983	1.673 418
27	1.352 634	1.398 511	1.494 800	1.597 457	1.706 886
28	1.367 852	1.415 992	1.517 222	1.625 413	1.741 024
29	1.383 240	1.433 692	1.539 981	1.653 858	1.775 845
30	1.398 801	1.451 613	1.563 080	1.682 800	1.811 362
31	1.414 538	1.469 759	1.586 526	1.712 249	1.847 589
32	1.430 451	1.488 131	1.610 324	1.742 213	1.884 541
33	1.446 544	1.506 732	1.634 479	1.772 702	1.922 231
34	1.462 818	1.525 566	1.658 996	1.803 725	1.960 676
35	1.479 274	1.544 636	1.683 881	1.835 290	1.999 890
36	1.495 916	1.563 944	1.709 140	1.867 407	2.039 887
37	1.512 745	1.583 493	1.734 777	1.900 087	2.080 685
38	1.529 764	1.603 287	1.760 798	1.933 338	2.122 299
39	1.546 973	1.623 328	1.787 210	1.967 172	2.164 745
40	1.564 377	1.643 619	1.814 018	2.001 597	2.208 040
41	1.581 976	1.664 165	1.841 229	2.036 625	2.252 200
42	1.599 773	1.684 967	1.868 847	2.072 266	2.297 244
43	1.617 771	1.706 029	1.896 880	2.108 531	2.343 189
44	1.635 971	1.727 354	1.925 333	2.145 430	2.390 053
45	1.654 375	1.748 946	1.954 213	2.182 975	2.437 854
46	1.672 987	1.770 808	1.983 526	2.221 177	2.486 611
47	1.691 808	1.792 943	2.013 279	2.260 048	2.536 344
48	1.710 841	1.815 355	2.043 478	2.299 599	2.587 070
49	1.730 088	1.838 047	2.074 130	2.339 842	2.638 812
50	1.749 552	1.861 022	2.105 242	2.380 789	2.691 588

$(1 + r)^n$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	1.769 234	1.884 285	2.136 821	2.422 453	2.745 420
52	1.789 138	1.907 839	2.168 873	2.464 846	2.800 328
53	1.809 266	1.931 687	2.201 406	2.507 980	2.856 335
54	1.829 620	1.955 833	2.234 428	2.551 870	2.913 461
55	1.850 203	1.980 281	2.267 944	2.596 528	2.971 731
56	1.871 018	2.005 034	2.301 963	2.641 967	3.031 165
57	1.892 067	2.030 097	2.336 493	2.688 202	3.091 789
58	1.913 353	2.055 473	2.371 540	2.735 245	3.153 624
59	1.934 878	2.081 167	2.407 113	2.783 112	3.216 697
60	1.956 645	2.107 181	2.443 220	2.831 816	3.281 031
61	1.978 657	2.133 521	2.479 868	2.881 373	3.346 651
62	2.000 917	2.160 190	2.517 066	2.931 797	3.413 584
63	2.023 428	2.187 193	2.554 822	2.983 104	3.481 856
64	2.046 191	2.214 532	2.593 144	3.035 308	3.551 493
65	2.069 211	2.242 214	2.632 042	3.088 426	3.622 523
66	2.092 489	2.270 242	2.671 522	3.142 473	3.694 974
67	2.116 030	2.298 620	2.711 595	3.197 466	3.768 873
68	2.139 835	2.327 353	2.752 269	3.253 422	3.844 251
69	2.163 908	2.356 444	2.793 553	3.310 357	3.921 136
70	2.188 252	2.385 900	2.835 456	3.368 288	3.999 558
71	2.212 870	2.415 724	2.877 988	3.427 233	4.079 549
72	2.237 765	2.445 920	2.921 158	3.487 210	4.161 140
73	2.262 940	2.476 494	2.964 975	3.548 236	4.244 363
74	2.288 398	2.507 450	3.009 450	3.610 330	4.329 250
75	2.314 142	2.538 794	3.054 592	3.673 511	4.415 835
76	2.340 177	2.570 529	3.100 411	3.737 797	4.504 152
77	2.366 504	2.602 660	3.146 917	3.803 209	4.594 235
78	2.393 127	2.635 193	3.194 120	3.869 765	4.686 120
79	2.420 049	2.668 133	3.242 032	3.937 486	4.779 842
80	2.447 275	2.701 485	3.290 663	4.006 392	4.875 439
81	2.474 807	2.735 254	3.340 023	4.076 504	4.972 948
82	2.502 648	2.769 444	3.390 123	4.147 843	5.072 407
83	2.530 803	2.804 062	3.440 975	4.220 430	5.173 855
84	2.559 275	2.839 113	3.492 590	4.294 287	5.277 332
85	2.588 067	2.874 602	3.544 978	4.369 437	5.382 879
86	2.617 182	2.910 534	3.598 153	4.445 903	5.490 536
87	2.646 626	2.946 916	3.652 125	4.523 706	5.600 347
88	2.676 400	2.983 753	3.706 907	4.602 871	5.712 354
89	2.706 510	3.021 049	3.762 511	4.683 421	5.826 601
90	2.736 958	3.058 813	3.818 949	4.765 381	5.943 133
91	2.767 749	3.097 048	3.876 233	4.848 775	6.061 996
92	2.798 886	3.135 761	3.934 376	4.933 629	6.183 236
93	2.830 373	3.174 958	3.993 392	5.019 967	6.306 900
94	2.862 215	3.214 645	4.053 293	5.107 816	6.433 038
95	2.894 415	3.254 828	4.114 092	5.197 203	6.561 699
96	2.926 977	3.295 513	4.175 804	5.288 154	6.692 933
97	2.959 906	3.336 707	4.238 441	5.380 697	6.826 792
98	2.993 205	3.378 416	4.302 017	5.474 859	6.963 328
99	3.026 878	3.420 646	4.366 547	5.570 669	7.102 594
100	3.060 930	3.463 404	4.432 046	5.668 156	7.244 646

$(1 + r)^n$

n	3%	4%	5%	6%	7%
1	1.030 000	1.040 000	1.050 000	1.060 000	1.070 000
2	1.060 900	1.081 600	1.102 500	1.123 600	1.144 900
3	1.092 727	1.124 864	1.157 625	1.191 016	1.225 043
4	1.125 509	1.169 859	1.215 506	1.262 477	1.310 796
5	1.159 274	1.216 653	1.276 282	1.338 226	1.402 552
6	1.194 052	1.265 319	1.340 096	1.418 519	1.500 730
7	1.229 874	1.315 932	1.407 100	1.503 630	1.605 781
8	1.266 770	1.368 569	1.477 455	1.593 848	1.718 186
9	1.304 773	1.423 312	1.551 328	1.689 479	1.838 459
10	1.343 916	1.480 244	1.628 895	1.790 848	1.967 151
11	1.384 234	1.539 454	1.710 339	1.898 299	2.104 852
12	1.425 761	1.601 032	1.795 856	2.012 196	2.252 192
13	1.468 534	1.665 074	1.885 649	2.132 928	2.409 845
14	1.512 590	1.731 676	1.979 932	2.260 904	2.578 534
15	1.557 967	1.800 944	2.078 928	2.396 558	2.759 032
16	1.604 706	1.872 981	2.182 875	2.540 352	2.952 164
17	1.652 848	1.947 900	2.292 018	2.692 773	3.158 815
18	1.702 433	2.025 817	2.406 619	2.854 339	3.379 932
19	1.753 506	2.106 849	2.526 950	3.025 600	3.616 528
20	1.806 111	2.191 123	2.653 298	3.207 135	3.869 684
21	1.860 295	2.278 768	2.785 963	3.399 564	4.140 562
22	1.916 103	2.369 919	2.925 261	3.603 537	4.430 402
23	1.973 587	2.464 716	3.071 524	3.819 750	4.740 530
24	2.032 794	2.563 304	3.225 100	4.048 935	5.072 367
25	2.093 778	2.665 836	3.386 355	4.291 871	5.427 433
26	2.156 591	2.772 470	3.555 673	4.549 383	5.807 353
27	2.221 289	2.883 369	3.733 456	4.822 346	6.213 868
28	2.287 928	2.998 703	3.920 129	5.111 687	6.648 838
29	2.356 566	3.118 651	4.116 136	5.418 388	7.114 257
30	2.427 262	3.243 398	4.321 942	5.743 491	7.612 255
31	2.500 080	3.373 133	4.538 039	6.088 101	8.145 113
32	2.575 083	3.508 059	4.764 941	6.453 387	8.715 271
33	2.652 335	3.648 381	5.003 189	6.840 590	9.325 340
34	2.731 905	3.794 316	5.253 348	7.251 025	9.978 114
35	2.813 862	3.946 089	5.516 015	7.686 087	10.676 581
36	2.898 278	4.103 933	5.791 816	8.147 252	11.423 942
37	2.985 227	4.268 090	6.081 407	8.636 087	12.223 618
38	3.074 783	4.438 813	6.385 477	9.154 252	13.079 271
39	3.167 027	4.616 366	6.704 751	9.703 507	13.994 820
40	3.262 038	4.801 021	7.039 989	10.285 718	14.974 458
41	3.359 899	4.993 061	7.391 988	10.902 861	16.022 670
42	3.460 696	5.192 784	7.761 588	11.557 033	17.144 257
43	3.564 517	5.400 495	8.149 667	12.250 455	18.344 355
44	3.671 452	5.616 515	8.557 150	12.985 482	19.628 460
45	3.781 596	5.841 176	8.985 008	13.764 611	21.002 452
46	3.895 044	6.074 823	9.434 258	14.590 487	22.472 623
47	4.011 895	6.317 816	9.905 971	15.465 917	24.045 707
48	4.132 252	6.570 528	10.401 270	16.393 872	25.728 907
49	4.256 219	6.833 349	10.921 333	17.377 504	27.529 930
50	4.383 906	7.106 683	11.467 400	18.420 154	29.457 025

$(1 + r)^n$

n	3%	4%	5%	6%	7%
51	4.515 423	7.390 951	12.040 770	19.525 364	31.519 017
52	4.650 886	7.686 589	12.642 808	20.696 885	33.725 348
53	4.790 412	7.994 052	13.274 949	21.938 698	36.086 122
54	4.934 125	8.313 814	13.938 696	23.255 020	38.612 151
55	5.082 149	8.646 367	14.635 631	24.650 322	41.315 001
56	5.234 613	8.992 222	15.367 412	26.129 341	44.207 052
57	5.391 651	9.351 910	16.135 783	27.697 101	47.301 545
58	5.553 401	9.725 987	16.942 572	29.358 927	50.612 653
59	5.720 003	10.115 026	17.789 701	31.120 463	54.155 539
60	5.891 603	10.519 627	18.679 186	32.987 691	57.946 427
61	6.068 351	10.940 413	19.613 145	34.966 952	62.002 677
62	6.250 402	11.378 029	20.593 802	37.064 969	66.342 864
63	6.437 914	11.833 150	21.623 493	39 288 868	70.986 865
64	6.631 051	12.306 476	22.704 667	41.646 200	75.955 945
65	6.829 983	12.798 735	23.839 901	44.144 972	81.272 861
66	7.034 882	13.310 685	25.031 896	46.793 670	86.961 962
67	7.245 929	13.843 112	26.283 490	49.601 290	93.049 299
68	7.463 307	14.396 836	27.597 665	52.577 368	99.562 750
69	7.687 206	14.972 710	28.977 548	55.732 010	106.532 142
70	7.917 822	15.571 618	30.426 426	59.075 930	113.989 392
71	8.155 357	16.194 483	31.947 747	62.620 486	121.968 650
72	8.400 017	16.842 262	33.545 134	66.377 715	130.506 455
73	8.652 018	17.515 953	35.222 391	70.360 378	139.641 907
74	8.911 578	18.216 591	36.983 510	74.582 001	149.416 840
75	9.178 926	18.945 255	38.832 686	79.056 921	159.876 019
76	9.454 293	19.703 065	40.774 320	83.800 336	171.067 341
77	9.737 922	20.491 187	42.813 036	88.828 356	183.042 055
78	10.030 060	21.310 835	44.953 688	94.158 058	195.854 998
79	10.330 962	22.163 268	47.201 372	99.807 541	209.564 848
80	10.640 891	23.049 799	49.561 441	105.795 993	224.234 388
81	10.960 117	23.971 791	52.039 531	112.143 753	239.930 795
82	11.288 921	24.930 663	54.641 489	118.872 378	256.725 950
83	11.627 588	25.927 889	57.373 563	126.004 721	274.696 767
84	11.976 416	26.965 005	60.242 241	133.565 004	293.925 541
85	12.335 709	28.043 605	63.254 353	141.578 904	314.500 328
86	12.705 780	29.165 349	66.417 071	150.073 639	336.515 351
87	13.086 953	30.331 963	69.737 925	159.078 057	360.071 426
88	13.479 562	31.545 242	73.224 821	168.622 741	385.276 426
89	13.883 949	32.807 051	76.886 062	178.740 105	412.245 776
90	14.300 467	34.119 333	80.730 365	189.464 511	441.102 980
91	14.729 481	35.484 107	84.766 883	200.832 382	471.980 188
92	15.171 366	36.903 471	89.005 227	212.882 325	505.018 802
93	15.626 507	38.379 610	93.455 489	225.655 264	540.370 118
94	16.095 302	39.914 794	98.128 263	239.194 580	578.196 026
95	16.578 161	41.511 386	103.034 676	253.546 255	618.669 748
96	17.075 506	43.171 841	108.186 410	268.759 030	661.976 630
97	17.587 771	44.898 715	113.595 731	284.884 572	708.314 994
98	18.115 404	46.694 664	119.275 517	301.977 646	757.897 044
99	18.658 866	48.562 450	125.239 293	320.096 305	810.949 837
100	19.218 632	50.504 948	131.501 258	339.302 084	867.716 326

Table IV

PRESENT VALUE: $(1 + r)^{-n}$

Note: The table gives six-place values of the present value of 1 for values of n from 1 to 100, and certain values of r from $\frac{5}{12}\%$ to 7%. In the formula, n is the number of conversion periods and r is the rate per period. The present value of S is found by multiplying the tabular value by S. To find the present value of \$125.41 due in 2 years if money is worth 2% compounded quarterly, we take $n = 8$ (2 times 4 periods per year) and $r = \frac{1}{2}\%$ (2% divided by 4). In the $\frac{1}{2}\%$ column opposite n equal to 8, we read 0.960885. The present value of \$125.41 under the stated conditions of time and rate is, therefore, 125.41 × 0.960885, or \$120.50.

$(1 + i)^{-n}$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	0.995 851	0.995 025	0.994 200	0.992 556	0.990 099
2	0.991 718	0.990 075	0.984 435	0.985 167	0.980 296
3	0.987 603	0.985 149	0.982 702	0.977 833	0.970 590
4	0.983 506	0.980 248	0.977 003	0.970 554	0.960 980
5	0.979 425	0.975 371	0.971 337	0.963 329	0.951 466
6	0.975 361	0.970 518	0.965 704	0.956 158	0.942 045
7	0.971 313	0.965 690	0.960 103	0.949 040	0.932 718
8	0.967 283	0.960 885	0.954 535	0.941 975	0.923 483
9	0.963 269	0.956 105	0.948 999	0.934 963	0.914 340
10	0.959 272	0.951 348	0.943 495	0.928 003	0.905 287
11	0.955 292	0.946 615	0.938 024	0.921 095	0.896 324
12	0.951 328	0.941 905	0.932 583	0.914 238	0.887 449
13	0.947 381	0.937 219	0.927 175	0.907 432	0.878 663
14	0.943 450	0.932 556	0.921 798	0.900 677	0.869 963
15	0.939 535	0.927 917	0.916 452	0.893 973	0.861 349
16	0.935 637	0.923 300	0.911 137	0.887 318	0.852 821
17	0.931 754	0.918 707	0.905 853	0.880 712	0.844 377
18	0.927 888	0.914 136	0.900 599	0.874 156	0.836 017
19	0.924 038	0.909 588	0.895 376	0.867 649	0.827 740
20	0.920 204	0.905 063	0.890 183	0.861 190	0.819 544
21	0.916 385	0.900 560	0.885 021	0.854 779	0.811 430
22	0.912 583	0.896 080	0.879 888	0.848 416	0.803 396
23	0.908 796	0.891 622	0.874 785	0.842 100	0.795 442
24	0.905 025	0.887 186	0.869 712	0.835 831	0.787 566
25	0.901 270	0.882 772	0.864 668	0.829 609	0.779 768
26	0.897 530	0.878 380	0.859 653	0.823 434	0.772 048
27	0.893 806	0.874 010	0.854 668	0.817 304	0.764 404
28	0.890 097	0.869 662	0.849 711	0.811 220	0.756 836
29	0.886 404	0.865 335	0.844 783	0.805 181	0.749 342
30	0.882 726	0.861 030	0.839 884	0.799 187	0.741 923
31	0.879 063	0.856 746	0.835 013	0.793 238	0.734 577
32	0.875 416	0.852 484	0.830 170	0.787 333	0.727 304
33	0.871 783	0.848 242	0.825 356	0.781 472	0.720 103
34	0.868 166	0.844 022	0.820 569	0.775 654	0.712 973
35	0.864 564	0.839 823	0.815 810	0.769 880	0.705 914
36	0.860 976	0.835 645	0.811 079	0.764 149	0.698 925
37	0.857 404	0.831 487	0.806 375	0.758 461	0.692 005
38	0.853 846	0.827 351	0.801 699	0.752 814	0.685 153
39	0.850 303	0.823 235	0.797 049	0.747 210	0.678 370
40	0.846 775	0.819 139	0.792 427	0.741 648	0.671 653
41	0.843 261	0.815 064	0.787 831	0.736 127	0.665 003
42	0.839 762	0.811 009	0.783 262	0.730 647	0.658 419
43	0.836 278	0.806 974	0.778 719	0.725 208	0.651 900
44	0.832 808	0.802 959	0.774 203	0.719 810	0.645 445
45	0.829 352	0.798 964	0.769 713	0.714 451	0.639 055
46	0.825 911	0.794 989	0.765 249	0.709 133	0.632 728
47	0.822 484	0.791 034	0.760 811	0.703 854	0.626 463
48	0.819 071	0.787 098	0.756 399	0.698 614	0.620 260
49	0.815 672	0.783 182	0.752 012	0.693 414	0.614 119
50	0.812 288	0.779 286	0.747 651	0.688 252	0.608 039

$(1 + r)^{-n}$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	0.808 917	0.775 409	0.743 315	0.683 128	0.602 019
52	0.805 561	0.771 551	0.739 004	0.678 043	0.596 058
53	0.802 218	0.767 713	0.734 718	0.672 995	0.590 156
54	0.798 890	0.763 893	0.730 457	0.667 986	0.584 313
55	0.795 575	0.760 093	0.726 221	0.663 013	0.578 528
56	0.792 274	0.756 311	0.722 009	0.658 077	0.572 800
57	0.788 986	0.752 548	0.717 822	0.653 178	0.567 129
58	0.785 712	0.748 804	0.713 659	0.648 316	0.561 514
59	0.782 452	0.745 079	0.709 520	0.643 490	0.555 954
60	0.779 205	0.741 372	0.705 405	0.638 700	0.550 450
61	0.775 972	0.737 684	0.701 314	0.633 945	0.545 000
62	0.772 752	0.734 014	0.697 247	0.629 226	0.539 604
63	0.769 546	0.730 362	0.693 203	0.624 542	0.534 261
64	0.766 353	0.726 728	0.689 183	0.619 893	0.528 971
65	0.763 173	0.723 113	0.685 186	0.615 278	0.523 734
66	0.760 006	0.719 515	0.681 212	0.610 698	0.518 548
67	0.756 853	0.715 935	0.677 262	0.606 152	0.513 414
68	0.753 712	0.712 374	0.673 334	0.601 639	0.508 331
69	0.750 585	0.708 829	0.669 429	0.597 161	0.503 298
70	0.747 470	0.705 303	0.665 546	0.592 715	0.498 315
71	0.744 369	0.701 794	0.661 687	0.588 303	0.493 381
72	0.741 280	0.698 302	0.657 849	0.583 924	0.488 496
73	0.738 204	0.694 828	0.654 034	0.579 577	0.483 659
74	0.735 141	0.691 371	0.650 241	0.575 262	0.478 871
75	0.732 091	0.687 932	0.646 470	0.570 980	0.474 129
76	0.729 053	0.684 509	0.642 721	0.566 730	0.469 435
77	0.726 028	0.681 104	0.638 993	0.562 511	0.464 787
78	0.723 015	0.677 715	0.635 287	0.558 323	0.460 185
79	0.720 015	0.674 343	0.631 603	0.554 167	0.455 629
80	0.717 028	0.670 988	0.627 940	0.550 042	0.451 118
81	0.714 052	0.667 650	0.624 298	0.545 947	0.446 651
82	0.711 090	0.664 329	0.620 678	0.541 883	0.442 229
83	0.708 139	0.661 023	0.617 078	0.537 849	0.437 851
84	0.705 201	0.657 735	0.613 499	0.533 845	0.433 515
85	0.702 275	0.654 462	0.609 941	0.529 871	0.429 223
86	0.699 361	0.651 206	0.606 404	0.525 927	0.424 974
87	0.696 459	0.647 967	0.602 887	0.522 012	0.420 766
88	0.693 569	0.644 743	0.599 391	0.518 126	0.416 600
89	0.690 691	0.641 535	0.595 914	0.514 269	0.412 475
90	0.687 825	0.638 344	0.592 458	0.510 440	0.408 391
91	0.684 971	0.635 168	0.589 022	0.506 641	0.404 348
92	0.682 129	0.632 008	0.585 606	0.502 869	0.400 344
93	0.679 298	0.628 863	0.582 210	0.499 126	0.396 380
94	0.676 480	0.625 735	0.578 834	0.495 410	0.392 456
95	0.673 673	0.622 622	0.575 477	0.491 722	0.388 570
96	0.670 877	0.619 524	0.572 139	0.488 062	0.384 723
97	0.668 094	0.616 442	0.568 821	0.484 428	0.380 914
98	0.665 321	0.613 375	0.565 522	0.480 822	0.377 142
99	0.662 561	0.610 323	0.562 242	0.477 243	0.373 408
100	0.659 812	0.607 287	0.558 982	0.473 690	0.369 711

$(1 + r)^{-n}$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	0.988 875	0.987 654	0.985 222	0.982 801	0.980 392
2	0.977 874	0.975 461	0.970 662	0.965 898	0.961 169
3	0.966 995	0.963 418	0.956 317	0.949 285	0.942 322
4	0.956 238	0.951 524	0.942 184	0.932 959	0.923 845
5	0.945 600	0.939 777	0.928 260	0.916 913	0.905 731
6	0.935 080	0.928 175	0.914 542	0.901 143	0.887 971
7	0.924 677	0.916 716	0.901 027	0.885 644	0.870 560
8	0.914 391	0.905 398	0.887 711	0.870 412	0.853 490
9	0.904 218	0.894 221	0.874 592	0.855 441	0.836 755
10	0.894 159	0.883 181	0.861 667	0.840 729	0.820 348
11	0.884 211	0.872 277	0.848 933	0.826 269	0.804 263
12	0.874 375	0.861 509	0.836 387	0.812 058	0.788 493
13	0.864 647	0.850 873	0.824 027	0.798 091	0.773 033
14	0.855 028	0.840 368	0.811 849	0.784 365	0.757 875
15	0.845 516	0.829 993	0.799 852	0.770 875	0.743 015
16	0.836 110	0.819 746	0.788 031	0.757 616	0.728 446
17	0.826 808	0.809 626	0.776 385	0.744 586	0.714 163
18	0.817 610	0.799 631	0.764 912	0.731 780	0.700 159
19	0.808 515	0.789 759	0.753 607	0.719 194	0.686 431
20	0.799 520	0.780 009	0.742 470	0.706 825	0.672 971
21	0.790 625	0.770 379	0.731 498	0.694 668	0.659 776
22	0.781 830	0.760 868	0.720 688	0.682 720	0.646 839
23	0.773 132	0.751 475	0.710 037	0.670 978	0.634 156
24	0.764 531	0.742 197	0.699 544	0.659 438	0.621 721
25	0.756 026	0.733 034	0.689 206	0.648 096	0.609 531
26	0.747 615	0.723 984	0.679 021	0.636 950	0.597 579
27	0.739 298	0.715 046	0.668 986	0.625 995	0.585 862
28	0.731 073	0.706 219	0.659 099	0.615 228	0.574 375
29	0.722 940	0.697 500	0.649 359	0.604 647	0.563 112
30	0.714 898	0.688 889	0.639 762	0.594 248	0.552 071
31	0.706 945	0.680 384	0.630 308	0.584 027	0.541 246
32	0.699 080	0.671 984	0.620 993	0.573 982	0.530 633
33	0.691 303	0.663 688	0.611 816	0.564 111	0.520 229
34	0.683 612	0.655 494	0.602 774	0.554 408	0.510 028
35	0.676 007	0.647 402	0.593 866	0.544 873	0.500 028
36	0.668 487	0.639 409	0.585 090	0.535 502	0.490 223
37	0.661 050	0.631 515	0.576 443	0.526 292	0.480 611
38	0.653 696	0.623 719	0.567 924	0.517 240	0.471 187
39	0.646 424	0.616 019	0.559 531	0.508 344	0.461 948
40	0.639 232	0.608 413	0.551 262	0.499 601	0.452 890
41	0.632 121	0.600 902	0.543 116	0.491 008	0.444 010
42	0.625 089	0.593 484	0.535 089	0.482 563	0.435 304
43	0.618 135	0.586 157	0.527 182	0.474 264	0.426 769
44	0.611 258	0.578 920	0.519 391	0.466 107	0.418 401
45	0.604 458	0.571 773	0.511 715	0.458 090	0.410 197
46	0.597 733	0.564 714	0.504 153	0.450 212	0.402 154
47	0.591 084	0.557 742	0.496 702	0.442 469	0.394 268
48	0.584 508	0.550 856	0.489 362	0.434 858	0.386 538
49	0.578 005	0.544 056	0.482 130	0.427 379	0.378 958
50	0.571 575	0.537 339	0.475 005	0.420 029	0.371 528

$(1 + r)^{-n}$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	0.565 216	0.530 705	0.467 985	0.412 805	0.364 243
52	0.558 928	0.524 153	0.461 069	0.405 705	0.357 101
53	0.552 710	0.517 682	0.454 255	0.398 727	0.350 099
54	0.546 562	0.511 291	0.447 542	0.391 869	0.343 234
55	0.540 481	0.504 979	0.440 928	0.385 130	0.336 504
56	0.534 468	0.498 745	0.434 412	0.378 506	0.329 906
57	0.528 523	0.492 587	0.427 992	0.371 996	0.323 437
58	0.522 643	0.486 506	0.421 667	0.365 598	0.317 095
59	0.516 829	0.480 500	0.415 435	0.359 310	0.310 878
60	0.511 079	0.474 568	0.409 296	0.353 130	0.304 782
61	0.505 393	0.468 709	0.403 247	0.347 057	0.298 806
62	0.499 771	0.462 922	0.397 288	0.341 088	0.292 947
63	0.494 211	0.457 207	0.391 417	0.335 221	0.287 203
64	0.488 713	0.451 563	0.385 632	0.329 456	0.281 572
65	0.483 276	0.445 988	0.379 933	0.323 790	0.276 051
66	0.477 900	0.440 482	0.374 318	0.318 221	0.270 638
67	0.472 583	0.435 044	0.368 787	0.312 748	0.265 331
68	0.467 326	0.429 673	0.363 337	0.307 369	0.260 129
69	0.462 127	0.424 368	0.357 967	0.302 082	0.255 028
70	0.456 986	0.419 129	0.352 677	0.296 887	0.250 028
71	0.451 902	0.413 955	0.347 465	0.291 781	0.245 125
72	0.446 874	0.408 844	0.342 330	0.286 762	0.240 319
73	0.441 903	0.403 797	0.337 271	0.281 830	0.235 607
74	0.436 987	0.398 811	0.332 287	0.276 983	0.230 987
75	0.432 126	0.393 888	0.327 376	0.272 219	0.226 458
76	0.427 318	0.389 025	0.322 538	0.267 537	0.222 017
77	0.422 564	0.384 222	0.317 771	0.262 936	0.217 664
78	0.417 863	0.379 479	0.313 075	0.258 414	0.213 396
79	0.413 215	0.374 794	0.308 448	0.253 969	0.209 212
80	0.408 618	0.370 167	0.303 890	0.249 601	0.205 110
81	0.404 072	0.365 597	0.299 399	0.245 308	0.201 088
82	0.399 577	0.361 083	0.294 975	0.241 089	0.197 145
83	0.395 131	0.356 625	0.290 615	0.236 943	0.193 279
84	0.390 736	0.352 223	0.286 321	0.232 868	0.189 490
85	0.386 389	0.347 874	0.282 089	0.228 862	0.185 774
86	0.382 090	0.343 580	0.277 920	0.224 926	0.182 132
87	0.377 840	0.339 338	0.273 813	0.221 058	0.178 560
88	0.373 636	0.335 148	0.269 767	0.217 256	0.175 059
89	0.369 480	0.331 011	0.265 780	0.213 519	0.171 627
90	0.365 369	0.326 924	0.261 852	0.209 847	0.168 261
91	0.361 304	0.322 888	0.257 982	0.206 238	0.164 962
92	0.357 285	0.318 902	0.254 170	0.202 691	0.161 728
93	0.353 310	0.314 965	0.250 414	0.199 204	0.158 556
94	0.349 380	0.311 076	0.246 713	0.195 778	0.155 448
95	0.345 493	0.307 236	0.243 067	0.192 411	0.152 400
96	0.341 649	0.303 443	0.239 475	0.189 102	0.149 411
97	0.337 849	0.299 697	0.235 936	0.185 850	0.146 482
98	0.334 090	0.295 997	0.232 449	0.182 653	0.143 609
99	0.330 373	0.292 342	0.229 014	0.179 512	0.140 794
100	0.326 698	0.288 733	0.225 629	0.176 424	0.138 033

$(1 + r)^{-n}$

n	3%	4%	5%	6%	7%
1	0.970 874	0.961 538	0.952 381	0.943 396	0.934 579
2	0.942 596	0.924 556	0.907 029	0.889 996	0.873 439
3	0.915 142	0.888 996	0.863 838	0.839 619	0.816 298
4	0.888 487	0.854 804	0.822 702	0.792 094	0.762 895
5	0.862 609	0.821 927	0.783 526	0.747 258	0.712 986
6	0.837 484	0.790 315	0.746 215	0.704 961	0.666 342
7	0.813 092	0.759 918	0.710 681	0.665 057	0.622 750
8	0.789 409	0.730 690	0.676 839	0.627 412	0.582 009
9	0.766 417	0.702 587	0.644 609	0.591 898	0.543 934
10	0.744 094	0.675 564	0.613 913	0.558 395	0.508 349
11	0.722 421	0.649 581	0.584 679	0.526 788	0.475 093
12	0.701 380	0.624 597	0.556 837	0.496 969	0.444 012
13	0.680 951	0.600 574	0.530 321	0.468 839	0.414 964
14	0.661 118	0.577 475	0.505 068	0.442 301	0.387 817
15	0.641 862	0.555 265	0.481 017	0.417 265	0.362 446
16	0.623 167	0.533 908	0.458 112	0.393 646	0.338 735
17	0.605 016	0.513 373	0.436 297	0.371 364	0.316 574
18	0.587 395	0.493 628	0.415 521	0.350 344	0.295 864
19	0.570 286	0.474 642	0.395 734	0.330 513	0.276 508
20	0.553 676	0.456 387	0.376 889	0.311 805	0.258 419
21	0.537 549	0.438 834	0.358 942	0.294 155	0.241 513
22	0.521 893	0.421 955	0.341 850	0.277 505	0.225 713
23	0.506 692	0.405 726	0.325 571	0.261 797	0.210 947
24	0.491 934	0.390 121	0.310 068	0.246 979	0.197 147
25	0.477 606	0.375 117	0.295 303	0.232 999	0.184 249
26	0.463 695	0.360 689	0.281 241	0.219 810	0.172 195
27	0.450 189	0.346 817	0.267 848	0.207 368	0.160 930
28	0.437 077	0.333 477	0.255 094	0.195 630	0.150 402
29	0.424 346	0.320 651	0.242 946	0.184 557	0.140 563
30	0.411 987	0.308 319	0.231 377	0.174 110	0.131 367
31	0.399 987	0.296 460	0.220 359	0.164 255	0.122 773
32	0.388 337	0.285 058	0.209 866	0.154 957	0.114 741
33	0.377 026	0.274 094	0.199 873	0.146 186	0.107 235
34	0.366 045	0.263 552	0.190 355	0.137 912	0.100 219
35	0.355 383	0.253 415	0.181 291	0.130 105	0.093 663
36	0.345 032	0.243 669	0.172 657	0.122 741	0.087 535
37	0.334 983	0.234 297	0.164 436	0.115 793	0.081 809
38	0.325 226	0.225 285	0.156 605	0.109 239	0.076 457
39	0.315 754	0.216 621	0.149 148	0.103 056	0.071 455
40	0.306 557	0.208 289	0.142 046	0.097 222	0.066 780
41	0.297 628	0.200 278	0.135 282	0.091 719	0.062 412
42	0.288 959	0.192 575	0.128 840	0.086 527	0.058 329
43	0.280 543	0.185 168	0.122 704	0.081 630	0.054 513
44	0.272 372	0.178 046	0.116 861	0.077 009	0.050 946
45	0.264 439	0.171 198	0.111 297	0.072 650	0.047 613
46	0.256 737	0.164 614	0.105 997	0.068 538	0.044 499
47	0.249 259	0.158 283	0.100 949	0.064 658	0.041 587
48	0.241 999	0.152 195	0.096 142	0.060 998	0.038 867
49	0.234 950	0.146 341	0.091 564	0.057 546	0.036 324
50	0.228 107	0.140 713	0.087 204	0.054 288	0.033 945

$(1 + r)^{-n}$

n	3%	4%	5%	6%	7%
51	0.221 463	0.135 301	0.083 051	0.051 215	0.031 727
52	0.215 013	0.130 097	0.079 096	0.048 316	0.029 651
53	0.208 750	0.125 093	0.075 330	0.045 582	0.027 711
54	0.202 670	0.120 282	0.071 743	0.043 001	0.025 899
55	0.196 767	0.115 656	0.068 326	0.040 567	0.024 204
56	0.191 036	0.111 207	0.065 073	0.038 271	0.022 621
57	0.185 472	0.106 930	0.061 974	0.036 105	0.021 141
58	0.180 070	0.102 817	0.059 023	0.034 061	0.019 758
59	0.174 825	0.098 863	0.056 212	0.032 133	0.018 465
60	0.169 733	0.095 060	0.053 536	0.030 314	0.017 257
61	0.164 789	0.091 404	0.050 986	0.028 598	0.016 128
62	0.159 990	0.087 889	0.048 558	0.026 980	0.015 073
63	0.155 330	0.084 508	0.046 246	0.025 453	0.014 087
64	0.150 806	0.081 258	0.044 044	0.024 012	0.013 166
65	0.146 413	0.078 133	0.041 946	0.022 653	0.012 304
66	0.142 149	0.075 128	0.039 949	0.021 370	0.011 499
67	0.138 009	0.072 238	0.038 047	0.020 161	0.010 747
68	0.133 989	0.069 460	0.036 235	0.019 020	0.010 044
69	0.130 086	0.066 788	0.034 509	0.017 943	0.009 387
70	0.126 297	0.064 219	0.032 866	0.016 927	0.008 773
71	0.122 619	0.061 749	0.031 301	0.015 969	0.008 199
72	0.119 047	0.059 374	0.029 811	0.015 065	0.007 662
73	0.115 580	0.057 091	0.028 391	0.014 213	0.007 161
74	0.112 214	0.054 895	0.027 039	0.013 408	0.006 693
75	0.108 945	0.052 784	0.025 752	0.012 649	0.006 255
76	0.105 772	0.050 754	0.024 525	0.011 933	0.005 846
77	0.102 691	0.048 801	0.023 357	0.011 258	0.005 463
78	0.099 700	0.046 924	0.022 245	0.010 620	0.005 106
79	0.096 796	0.045 120	0.021 186	0.010 019	0.004 772
80	0.093 977	0.043 384	0.020 177	0.009 452	0.004 460
81	0.091 240	0.041 716	0.019 216	0.008 917	0.004 168
82	0.088 582	0.040 111	0.018 301	0.008 412	0.003 895
83	0.086 002	0.038 569	0.017 430	0.007 936	0.003 640
84	0.083 497	0.037 085	0.016 600	0.007 487	0.003 402
85	0.081 065	0.035 659	0.015 809	0.007 063	0.003 180
86	0.078 704	0.034 287	0.015 056	0.006 663	0.002 972
87	0 076 412	0.032 969	0.014 339	0.006 286	0 002 777
88	0.074 186	0.031 701	0.013 657	0.005 930	0.002 596
89	0.072 026	0.030 481	0 013 006	0.005 595	0.002 426
90	0.069 928	0.029 309	0.012 387	0.005 278	0.002 267
91	0.067 891	0.028 182	0.011 797	0.004 979	0.002 119
92	0.065 914	0.027 098	0.011 235	0.004 697	0.001 980
93	0.063 994	0.026 056	0.010 700	0.004 432	0.001 851
94	0.062 130	0.025 053	0.010 191	0.004 181	0.001 730
95	0.060 320	0.024 090	0.009 705	0.003 944	0.001 616
96	0.058 563	0.023 163	0.009 243	0.003 721	0.001 511
97	0.056 858	0.022 272	0.008 803	0.003 510	0.001 412
98	0.055 202	0.021 416	0.008 384	0.003 312	0.001 319
99	0.053 594	0.020 592	0.007 985	0.003 124	0.001 233
100	0.052 033	0.019 800	0.007 604	0.002 947	0.001 152

Table V

AMOUNT OF ANNUITY: $\frac{(1+r)^n - 1}{r}$

Note: The table gives six-place values of the amount of an annuity of 1 per period for values of n from 1 to 100, and certain values of r from $\frac{5}{12}$% to 7%. In the formula, n is the number of periods and r is the rate per period. The amount of an annuity of R dollars per period is found by multiplying the tabular value by R. For example, to find the amount of an annuity of $500 per year for 8 years at 4%, we read in the 4% column opposite an n of 8 the amount 9.214226 and multiply by 500 to obtain $4,607.11.

$$\frac{(1 + r)^n - 1}{r}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
2	2.004 167	2.005 000	2.005 833	2.007 500	2.010 000
3	3.012 517	3.015 025	3.017 534	3.022 556	3.030 100
4	4.025 070	4.030 100	4.035 136	4.045 225	4.060 401
5	5.041 841	5.050 251	5.058 675	5.075 565	5.101 005
6	6.062 848	6.075 502	6.088 184	6.113 631	6.152 015
7	7.088 110	7.105 879	7.123 698	7.159 484	7.213 535
8	8.117 644	8.141 409	8.165 253	8.213 180	8.285 671
9	9.151 467	9.182 116	9.212 883	9.274 779	9.368 527
10	10.189 599	10.228 026	10.266 625	10.344 339	10.462 213
11	11.232 055	11.279 167	11.326 514	11.421 922	11.566 835
12	12.278 855	12.335 562	12.392 585	12.507 586	12.682 503
13	13.330 017	13.397 240	13.464 875	13.601 393	13.809 328
14	14.385 559	14.464 226	14.543 420	14.703 404	14.947 421
15	15.445 499	15.536 548	15.628 257	15.813 679	16.096 896
16	16.509 855	16.614 230	16.719 422	16.932 282	17.257 864
17	17.578 646	17.697 301	17.816 952	18.059 274	18.430 443
18	18.651 891	18.785 788	18.920 884	19.194 718	19.614 748
19	19.729 607	19.879 717	20.031 256	20.338 679	20.810 895
20	20.811 814	20.979 115	21.148 105	21.491 219	22.019 004
21	21.898 529	22.084 011	22.271 469	22.652 403	23.239 194
22	22.989 773	23.194 431	23.401 386	23.822 296	24.471 586
23	24.085 564	24.310 403	24.537 894	25.000 963	25.716 302
24	25.185 921	25.431 955	25.681 032	26.188 471	26.973 465
25	26.290 862	26.559 115	26.830 838	27.384 884	28.243 200
26	27.400 407	27.691 911	27.987 351	28.590 271	29.525 631
27	28.514 575	28.830 370	29.150 610	29.804 698	30.820 888
28	29.633 386	29.974 522	30.320 656	31.028 233	32.129 097
29	30.756 859	31.124 395	31.497 526	32.260 945	33.450 388
30	31.885 012	32.280 017	32.681 262	33.502 902	34.784 892
31	33.017 866	33.441 417	33.871 902	34.754 174	36.132 740
32	34.155 441	34.608 624	35.069 488	36.014 830	37.494 068
33	35.297 755	35.781 667	36.274 060	37.284 941	38.869 009
34	36.444 829	36.960 575	37.485 659	38.564 578	40.257 699
35	37.596 683	38.145 378	38.704 325	39.853 813	41.660 276
36	38.753 336	39.336 105	39.930 101	41.152 716	43.076 878
37	39.914 808	40.532 785	41.163 026	42.461 361	44.507 647
38	41.081 119	41.735 449	42.403 144	43.779 822	45.952 724
39	42.252 291	42.944 127	43.650 496	45.108 170	47.412 251
40	43.428 342	44.158 847	44.905 124	46.446 482	48.886 373
41	44.609 293	45.379 642	46.167 070	47.794 830	50.375 237
42	45.795 165	46.606 540	47.436 378	49.153 291	51.878 989
43	46.985 979	47.839 572	48.713 090	50.521 941	53.397 779
44	48.181 754	49.078 770	49.997 250	51.900 856	54.931 759
45	49.382 511	50.324 164	51.288 900	53.290 112	56.481 075
46	50.588 271	51 575 785	52.588 086	54.689 788	58.045 885
47	51.799 056	52.833 664	53.894 850	56.099 961	59.626 344
48	53.014 885	54.097 832	55.209 236	57.520 711	61.222 608
49	54.235 781	55.368 321	56.531 290	58.952 116	62.834 834
50	55.461 763	56.645 163	57.861 056	60.394 257	64.463 182

$$\frac{(1+r)^n - 1}{r}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	56.692 854	57.928 389	59.198 579	61.847 214	66.107 814
52	57.929 074	59.218 031	60.543 904	63.311 068	67.768 892
53	59.170 445	60.514 121	61.897 077	64.785 901	69.446 581
54	60.416 989	61.816 692	63.258 143	66.271 796	71.141 047
55	61.668 726	63.125 775	64.627 149	67.768 834	72.852 457
56	62.925 679	64.441 404	66.004 140	69.277 100	74.580 982
57	64.187 869	65.763 611	67.389 165	70.796 679	76.326 792
58	65.455 319	67.092 429	68.782 268	72.327 654	78.090 060
59	66.728 049	68.427 891	70.183 498	73.870 111	79.870 960
60	68.006 083	69.770 031	71.592 902	75.424 137	81.669 670
61	69.289 442	71.118 881	73.010 527	76.989 818	83.486 367
62	70.578 148	72.474 475	74.436 422	78.567 242	85.321 230
63	71.872 223	73.836 847	75.870 634	80.156 496	87.174 443
64	73.171 691	75.206 032	77.313 213	81.757 670	89.046 187
65	74.476 573	76.582 062	78.764 207	83.370 852	90.936 649
66	75.786 892	77.964 972	80.223 664	84.996 134	92.846 015
67	77.102 671	79.354 797	81.691 636	86.633 605	94.774 475
68	78.423 932	80.751 571	83.168 170	88.283 357	96.722 220
69	79.750 698	82.155 329	84.653 318	89.945 482	98.689 442
70	81.082 993	83.566 105	86.147 129	91.620 073	100.676 337
71	82.420 838	84.983 936	87.649 654	93.307 223	102.683 100
72	83.764 259	86.408 856	89.160 944	95.007 028	104.709 931
73	85.113 276	87.840 900	90.681 049	96.719 580	106.757 031
74	86.467 915	89.280 104	92.210 022	98.444 977	108.824 601
75	87.828 198	90.726 505	93.747 914	100.183 314	110.912 847
76	89.194 149	92.180 138	95.294 777	101.934 689	113.021 975
77	90.565 791	93.641 038	96.850 663	103.699 199	115.152 195
78	91.943 149	95.109 243	98.415 625	105.476 943	117.303 717
79	93.326 245	96.584 790	99.989 716	107.268 021	119.476 754
80	94.715 104	98.067 714	101.572 989	109.072 531	121.671 522
81	96.109 751	99.558 052	103.165 498	110.890 575	123.888 237
82	97.510 208	101.055 842	104.767 297	112.722 254	126.127 119
83	98.916 500	102.561 122	106.378 440	114.567 671	128.388 390
84	100.328 653	104.073 927	107.998 981	116.426 928	130.672 274
85	101.746 689	105.594 297	109.628 975	118.300 130	132.978 997
86	103.170 633	107.122 268	111.268 477	120.187 381	135.308 787
87	104.600 511	108.657 880	112.917 543	122.088 787	137.661 875
88	106.036 346	110.201 169	114.576 229	124.004 453	140.038 494
89	107.478 164	111.752 175	116.244 590	125.934 486	142.438 879
90	108.925 990	113.310 936	117.922 684	127.878 995	144.863 267
91	110.379 848	114.877 490	119.610 566	129.838 087	147.311 900
92	111.839 764	116.451 878	121.308 294	131.811 873	149.785 019
93	113.305 763	118.034 137	123.015 926	133.800 462	152.282 869
94	114.777 871	119.624 308	124.733 519	135.803 965	154.805 698
95	116.256 112	121.222 430	126.461 131	137.822 495	157.353 755
96	117.740 512	122.828 542	128.198 821	139.856 164	159.927 293
97	119.231 098	124.442 684	129.946 647	141.905 085	162.526 565
98	120.727 894	126.064 898	131.704 670	143.969 373	165.151 831
99	122.230 927	127.695 222	133.472 947	146.049 143	167.803 349
100	123.740 222	129.333 698	135.251 539	148.144 512	170.481 383

$$\frac{(1 + r)^n - 1}{r}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
2	2.011 250	2.012 500	2.015 000	2.017 500	2.020 000
3	3.033 877	3.037 656	3.045 225	3.052 806	3.060 400
4	4.068 008	4.075 627	4.090 903	4.106 230	4.121 608
5	5.113 773	5.126 572	5.152 267	5.178 089	5.204 040
6	6.171 303	6.190 654	6.229 551	6.268 706	6.308 121
7	7.240 730	7.268 038	7.322 994	7.378 408	7.434 283
8	8.322 188	8.358 888	8.432 839	8.507 530	8.582 969
9	9.415 813	9.463 374	9.559 332	9.656 412	9.754 628
10	10.521 741	10.581 666	10.702 722	10.825 399	10.949 721
11	11.640 110	11.713 937	11.863 262	12.014 844	12.168 715
12	12.771 061	12.860 361	13.041 211	13.225 104	13.412 090
13	13.914 736	14.021 116	14.236 830	14.456 543	14.680 332
14	15.071 277	15.196 380	15.450 382	15.709 533	15.973 938
15	16.240 828	16.386 335	16.682 138	16.984 449	17.293 417
16	17.423 538	17.591 164	17.932 370	18.281 677	18.639 285
17	18.619 553	18.811 053	19.201 355	19.601 607	20.012 071
18	19.829 023	20.046 192	20.489 376	20.944 635	21.412 312
19	21.052 099	21.296 769	21.796 716	22.311 166	22.840 559
20	22.288 935	22.562 979	23.123 667	23.701 611	24.297 370
21	23.539 686	23.845 016	24.470 522	25.116 389	25.783 317
22	24.804 507	25.143 078	25.837 580	26.555 926	27.298 984
23	26.083 558	26.457 367	27.225 144	28.020 655	28.844 963
24	27.376 998	27.788 084	28.633 521	29.511 016	30.421 862
25	28.684 989	29.135 435	30.063 024	31.027 459	32.030 300
26	30.007 695	30.499 628	31.513 969	32.570 440	33.670 906
27	31.345 282	31.880 873	32.986 678	34.140 422	35.344 324
28	32.697 916	33.279 384	34.481 479	35.737 880	37.051 210
29	34.065 768	34.695 377	35.998 701	37.363 293	38.792 235
30	35.449 008	36.129 069	37.538 681	39.017 150	40.568 079
31	36.847 809	37.580 682	39.101 762	40.699 950	42.379 441
32	38.262 347	39.050 441	40.688 288	42.412 200	44.227 030
33	39.692 798	40.538 571	42.298 612	44.154 413	46.111 570
34	41.139 342	42.045 303	43.933 092	45.927 115	48.033 802
35	42.602 160	43.570 870	45.592 088	47.730 840	49.994 478
36	44.081 434	45.115 505	47.275 969	49.566 129	51.994 367
37	45.577 350	46.679 449	48.985 109	51.433 537	54.034 255
38	47.090 095	48.262 942	50.719 885	53.333 624	56.114 940
39	48.619 859	49.866 229	52.480 684	55.266 962	58.237 238
40	50.166 832	51.489 557	54.267 894	57.234 134	60.401 983
41	51.731 209	53.133 177	56.081 912	59.235 731	62.610 023
42	53.313 185	54.797 341	57.923 141	61.272 357	64.862 223
43	54.912 959	56.482 308	59.791 988	63.344 623	67.159 468
44	56.530 730	58.188 337	61.688 868	65.453 154	69.502 657
45	58.166 700	59.915 691	63.614 201	67.598 584	71.892 710
46	59.821 076	61.664 637	65.568 414	69.781 559	74.330 564
47	61.494 063	63.435 445	67.551 940	72.002 736	76.817 176
48	63.185 871	65.228 388	69.565 219	74.262 784	79.353 519
49	64.896 712	67.043 743	71.608 698	76.562 383	81.940 590
50	66.626 800	68.881 790	73.682 828	78.902 225	84.579 401

$$\frac{(1 + r)^n - 1}{r}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	68.376 352	70.742 812	75.788 070	81.283 014	87.270 989
52	70.145 585	72.627 097	77.924 892	83.705 466	90.016 409
53	71.934 723	74.534 936	80.093 765	86.170 312	92.816 737
54	73.743 989	76.466 623	82.295 171	88.678 292	95.673 072
55	75.573 609	78.422 456	84.529 599	91.230 163	98.586 534
56	77.423 812	80.402 736	86.797 543	93.826 690	101.558 264
57	79.294 830	82.407 771	89.099 506	96.468 658	104.589 430
58	81.186 897	84.437 868	91.435 999	99.156 859	107.681 218
59	83.100 249	86.493 341	93.807 539	101.892 104	110.834 843
60	85.035 127	88.574 508	96.214 652	104.675 216	114.051 539
61	86.991 772	90.681 689	98.657 871	107.507 032	117.332 570
62	88.970 430	92.815 210	101.137 740	110.388 405	120.679 222
63	90.971 347	94.975 400	103.654 806	113.320 202	124.092 806
64	92.994 775	97.162 593	106.209 628	116.303 306	127.574 662
65	95.040 966	99.377 125	108.802 772	119.338 614	131.126 155
66	97.110 177	101.619 339	111.434 814	122.427 039	134.748 679
67	99.202 666	103.889 581	114.106 336	125.569 513	138.443 652
68	101.318 696	106.188 201	116.817 931	128.766 979	142.212 525
69	103.458 532	108.515 553	119.570 200	132.020 401	146.056 776
70	105.622 440	110.871 998	122.363 753	135.330 758	149.977 911
71	107.810 692	113.257 898	125.199 209	138.699 047	153 977 469
72	110.023 563	115.673 621	128.077 197	142.126 280	158 057 019
73	112.261 328	118.119 542	130.998 355	145.613 490	162.218 159
74	114.524 268	120.596 036	133.963 331	149.161 726	166.462 522
75	116.812 666	123.103 486	136.972 781	152.772 056	170.791 773
76	119.126 808	125.642 280	140.027 372	156.445 567	175.207 608
77	121.466 985	128.212 809	143.127 783	160.183 364	179.711 760
78	123.833 488	130.815 469	146.274 700	163.986 573	184.305 996
79	126.226 615	133.450 662	149.468 820	167.856 338	188.992 115
80	128.646 665	136.118 795	152.710 852	171.793 824	193.771 958
81	131.093 940	138.820 280	156.001 515	175.800 216	198.647 397
82	133.568 746	141.555 534	159.341 538	179.876 720	203.620 345
83	136.071 395	144.324 978	162.731 661	184.024 563	208.692 752
84	138.602 198	147.129 040	166.172 636	188.244 992	213.866 607
85	141.161 473	149.968 153	169.665 226	192.539 280	219.143 939
86	143.749 539	152.842 755	173.210 204	196.908 717	224.526 818
87	146.366 722	155.753 289	176.808 357	201.354 620	230.017 354
88	149.013 347	158.700 206	180.460 482	205.878 326	235.617 701
89	151.689 747	161.683 958	184.167 390	210.481 196	241.330 055
90	154.396 257	164.705 008	187.929 900	215.164 617	247.156 656
91	157.133 215	167.763 820	191.748 849	219.929 998	253.099 789
92	159.900 964	170.860 868	195.625 082	224.778 773	259.161 785
93	162.699 849	173.996 629	199.559 458	229.712 401	265.345 021
94	165.530 223	177.171 587	203.552 850	234.732 369	271.651 921
95	168.392 438	180.386 232	207.606 142	239.840 185	278.084 960
96	171.286 853	183.641 059	211.720 235	245.037 388	284.646 659
97	174.213 830	186.936 573	215.896 038	250.325 542	291.339 592
98	177.173 735	190.273 280	220.134 479	255.706 239	298.166 384
99	180.166 940	193.651 696	224.436 496	261.181 099	305.129 712
100	183.193 818	197.072 342	228.803 043	266.751 768	312.232 306

$$\frac{(1+r)^n - 1}{r}$$

n	3%	4%	5%	6%	7%
1	1.000 000	1.000 000	1.000 000	1.000 000	1.000 000
2	2.030 000	2.040 000	2.050 000	2.060 000	2.070 000
3	3.090 900	3.121 600	3.152 500	3.183 600	3.214 900
4	4.183 627	4.246 464	4.310 125	4.374 616	4.439 943
5	5.309 136	5.416 323	5.525 631	5.637 093	5.750 739
6	6.468 410	6.632 975	6.801 913	6.975 319	7.153 291
7	7.662 462	7.898 294	8.142 008	8.393 838	8.654 021
8	8.892 336	9.214 226	9.549 109	9.897 468	10.259 803
9	10.159 106	10.582 795	11.026 564	11.491 316	11.977 989
10	11.463 879	12.006 107	12.577 893	13.180 795	13.816 448
11	12.807 796	13.486 351	14.206 787	14.971 643	15.783 599
12	14.192 030	15.025 805	15.917 127	16.869 941	17.888 451
13	15.617 790	16.626 838	17.712 983	18.882 138	20.140 643
14	17.086 324	18.291 911	19.598 632	21.015 066	22.550 488
15	18.598 914	20.023 588	21.578 564	23.275 970	25.129 022
16	20.156 881	21.824 531	23.657 492	25.672 528	27.888 054
17	21.761 588	23.697 512	25.840 366	28.212 880	30.840 217
18	23.414 435	25.645 413	28.132 385	30.905 653	33.999 033
19	25.116 868	27.671 229	30.539 004	33.759 992	37.378 965
20	26.870 374	29.778 079	33.065 954	36.785 591	40.995 492
21	28.676 486	31.969 202	35.719 252	39.992 727	44.865 177
22	30.536 780	34.247 970	38.505 214	43.392 290	49.005 739
23	32.452 884	36.617 889	41.430 475	46.995 828	53.436 141
24	34.426 470	39.082 604	44.501 999	50.815 577	58.176 671
25	36.459 264	41.645 908	47.727 099	54.864 512	63.249 038
26	38.553 042	44.311 745	51.113 454	59.156 383	68.676 470
27	40.709 634	47.084 214	54.669 127	63.705 766	74.483 823
28	42.930 923	49.967 583	58.402 583	68.528 112	80.697 691
29	45.218 850	52.966 286	62.322 712	73.639 798	87.346 529
30	47.575 416	56.084 938	66.438 848	79.058 186	94.460 786
31	50.002 678	59.328 335	70.760 790	84.801 677	102.073 041
32	52.502 759	62.701 469	75.298 829	90.889 778	110.218 154
33	55.077 841	66.209 527	80.063 771	97.343 165	118.933 425
34	57.730 177	69.857 909	85.066 959	104.183 755	128.258 765
35	60.462 082	73.652 225	90.320 307	111.434 780	138.236 878
36	63.275 944	77.598 314	95.836 323	119.120 867	148.913 460
37	66.174 223	81.702 246	101.628 139	127.268 119	160.337 402
38	69.159 449	85.970 336	107.709 546	135.904 206	172.561 020
39	72.234 233	90.409 150	114.095 023	145.058 458	185.640 292
40	75.401 260	95.025 516	120.799 774	154.761 966	199.635 112
41	78.663 298	99.826 536	127.839 763	165.047 684	214.609 570
42	82.023 196	104.819 598	135.231 751	175.950 545	230.632 240
43	85.483 892	110.012 382	142.993 333	187.507 577	247.776 496
44	89.048 409	115.412 877	151.143 006	199.758 032	266.120 851
45	92.719 861	121.029 392	159.700 156	212.743 514	285.749 311
46	96.501 457	126.870 568	168.685 164	226.508 125	306.751 763
47	100.396 501	132.945 390	178.119 422	241.098 612	329.224 386
48	104.408 396	139.263 206	188.025 393	256.564 529	353.270 093
49	108.540 648	145.833 734	198.426 663	272.958 401	378.999 000
50	112.796 867	152.667 084	209.347 996	290.335 905	406.528 929

$$\frac{(1+r)^n - 1}{r}$$

n	3%	4%	5%	6%	7%
51	117.180 773	159.773 767	220.815 396	308.756 059	435.985 955
52	121.696 197	167.164 718	232.856 165	328.281 422	467.504 971
53	126.347 082	174.851 306	245.498 974	348.978 308	501.230 319
54	131.137 495	182.845 359	258.773 922	370.917 006	537.316 442
55	136.071 620	191.159 173	272.712 618	394.172 027	575.928 593
56	141.153 768	199.805 540	287.348 249	418.822 348	617.243 594
57	146.388 381	208.797 762	302.715 662	444.951 689	661.450 646
58	151.780 033	218.149 672	318.851 445	472.648 790	708.752 191
59	157.333 434	227.875 659	335.794 017	502.007 718	759.364 844
60	163.053 437	237.990 685	353.583 718	533.128 181	813.520 383
61	168.945 040	248.510 313	372.262 904	566.115 872	871.466 810
62	175.013 391	259.450 725	391.876 049	601.082 824	933.469 487
63	181.263 793	270.828 754	412.469 851	638.147 793	999.812 351
64	187.701 707	282.661 904	434.093 344	677.436 661	1070.799 216
65	194.332 758	294.968 380	456.798 011	719.082 861	1146.755 161
66	201.162 741	307.767 116	480.637 912	763.227 832	1228.028 022
67	208.197 623	321.077 800	505.669 807	810.021 502	1314.989 983
68	215.443 551	334.920 912	531.953 298	859.622 792	1408.039 282
69	222.906 858	349.317 749	559.550 963	912.200 160	1507.602 032
70	230.594 064	364.290 459	588.528 511	967.932 170	1614.134 174
71	238.511 886	379.862 077	618.954 936	1027.008 100	1728.123 566
72	246.667 242	396.056 560	650.902 683	1089.628 586	1850.092 216
73	255.067 259	412.898 823	684.447 817	1156.006 301	1980.598 671
74	263.719 277	430.414 776	719.670 208	1226.366 679	2120.240 578
75	272.630 856	448.631 367	756.653 718	1300.948 680	2269.657 419
76	281.809 781	467.576 621	795.486 404	1380.005 601	2429.533 438
77	291.264 075	487.279 686	836.260 725	1463.805 937	2600.600 779
78	301.001 997	507.770 873	879.073 761	1552.634 293	2783.642 833
79	311.032 057	529.081 708	924.027 449	1646.792 350	2979.497 831
80	321.363 019	551.244 977	971.228 821	1746.599 891	3189.062 680
81	332.003 909	574.294 776	1020.790 262	1852.395 885	3413.297 067
82	342.964 026	598.266 567	1072.829 776	1964.539 638	3653.227 862
83	354.252 947	623.197 230	1127.471 264	2083.412 016	3909.953 812
84	365.880 536	649.125 119	1184.844 828	2209.416 737	4184.650 579
85	377.856 952	676.090 123	1245.087 069	2342.981 741	4478.576 120
86	390.192 660	704.133 728	1308.341 422	2484.560 646	4793.076 448
87	402.898 440	733.299 078	1374.758 493	2634.634 285	5129.591 799
88	415.985 393	763.631 041	1444.496 418	2793.712 342	5489.663 225
89	429.464 955	795.176 282	1517.721 239	2962.335 082	5874.939 651
90	443.348 904	827.983 334	1594.607 301	3141.075 187	6287.185 427
91	457.649 371	862.102 667	1675.337 666	3330.539 698	6728.288 407
92	472.378 852	897.586 774	1760.104 549	3531.372 080	7200.268 595
93	487.550 217	934.490 245	1849.109 777	3744.254 405	7705.287 397
94	503.176 724	972.869 854	1942.565 266	3969.909 669	8245.657 515
95	519.272 026	1012.784 648	2040.693 529	4209.104 250	8823.853 541
96	535.850 186	1054.296 034	2143.728 205	4462.650 505	9442.523 288
97	552.925 692	1097.467 876	2251.914 616	4731.409 535	10104.499 919
98	570.513 463	1142.366 591	2365.510 346	5016.294 107	10812.814 913
99	588.628 867	1189.061 254	2484.785 864	5318.271 753	11570.711 957
100	607.287 733	1237.623 705	2610.025 157	5638.368 059	12381.661 794

Table VI

PRESENT VALUE OF ANNUITY: $\frac{1 - (1 + r)^{-n}}{r}$

Note: The table gives six-place values of the present value of an annuity of 1 per period for values of n from 1 to 100, and certain values of r from $\frac{5}{12}$% to 7%. In the formula, n is the number of periods and r is the rate per period. The present value of R dollars per period is found by multiplying the tabular value by R. For example, to find the present value of an annuity of $500 a year for 6 years at 4%, we read in the 4% column opposite an n of 6 the value 5.242137 and multiply by 500 to obtain $2,621.07.

$$\frac{1-(1+r)^{-n}}{r}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
1	0.995 851	0.995 025	0.994 200	0.992 556	0.990 099
2	1.987 569	1.985 099	1.982 635	1.977 723	1.970 395
3	2.975 173	2.970 248	2.965 337	2.955 556	2.940 985
4	3.958 678	3.950 496	3.942 340	3.926 110	3.901 966
5	4.938 103	4.925 866	4.913 677	4.889 440	4.853 431
6	5.913 463	5.896 384	5.879 381	5.845 598	5.795 476
7	6.884 777	6.862 074	6.839 484	6.794 638	6.728 195
8	7.852 060	7.822 959	7.794 019	7.736 613	7.651 678
9	8.815 329	8.779 064	8.743 018	8.671 576	8.566 018
10	9.774 602	9.730 412	9.686 513	9.599 580	9.471 305
11	10.729 894	10.677 027	10.624 537	10.520 675	10.367 628
12	11.681 222	11.618 932	11.557 120	11.434 913	11.255 077
13	12.628 603	12.556 151	12.484 295	12.342 345	12.133 740
14	13.572 053	13.488 708	13.406 093	13.243 022	13.003 703
15	14.511 588	14.416 625	14.322 545	14.136 995	13.865 053
16	15.447 224	15.339 925	15.233 682	15.024 313	14.717 874
17	16.378 978	16.258 631	16.139 534	15.905 025	15.562 251
18	17.306 867	17.172 768	17.040 133	16.779 181	16.398 269
19	18.230 904	18.082 356	17.935 510	17.646 830	17.226 008
20	19.151 108	18.987 419	18.825 693	18.508 020	18.045 553
21	20.067 494	19.887 979	19.710 714	19.362 799	18.856 983
22	20.980 077	20.784 059	20.590 602	20.211 215	19.660 379
23	21.888 873	21.675 681	21.465 387	21.053 315	20.455 821
24	22.793 898	22.562 866	22.335 099	21.889 146	21.243 387
25	23.695 169	23.445 638	23.199 767	22.718 755	22.023 156
26	24.592 699	24.324 018	24.059 421	23.542 189	22.795 204
27	25.486 505	25.198 028	24.914 089	24.359 493	23.559 608
28	26.376 603	26.067 689	25.763 800	25.170 713	24.316 443
29	27.263 007	26.933 024	26.608 583	25.975 893	25.065 785
30	28.145 733	27.794 054	27.448 467	26.775 080	25.807 708
31	29.024 796	28.650 800	28.283 480	27.568 318	26.542 285
32	29.900 212	29.503 284	29.113 650	28.355 650	27.269 589
33	30.771 995	30.351 526	29.939 006	29.137 122	27.989 693
34	31.640 161	31.195 548	30.759 575	29.912 776	28.702 666
35	32.504 725	32.035 371	31.575 385	30.682 656	29.408 580
36	33.365 701	32.871 016	32.386 464	31.446 805	30.107 505
37	34.223 105	33.702 504	33.192 840	32.205 266	30.799 510
38	35.076 951	34.529 854	33.994 538	32.958 080	31.484 663
39	35.927 254	35.353 089	34.791 587	33.705 290	32.163 033
40	36.774 029	36.172 228	35.584 014	34.446 938	32.834 686
41	37.617 290	36.987 291	36.371 845	35.183 065	33.499 689
42	38.457 053	37.798 300	37.155 107	35.913 713	34.158 108
43	39.293 330	38.605 274	37.933 826	36.638 921	34.810 008
44	40.126 138	39.408 232	38.708 029	37.358 730	35.455 454
45	40.955 490	40.207 196	39.477 742	38.073 181	36.094 508
46	41.781 401	41.002 185	40.242 991	38.782 314	36.727 236
47	42.603 885	41.793 219	41.003 803	39.486 168	37.353 699
48	43.422 956	42.580 318	41.760 201	40.184 782	37.973 959
49	44.238 628	43.363 500	42.512 213	40.878 195	38.588 079
50	45.050 916	44.142 786	43.259 864	41.566 447	39.196 118

$$\frac{1-(1+r)^{-n}}{r}$$

n	$\frac{5}{12}\%$	$\frac{1}{2}\%$	$\frac{7}{12}\%$	$\frac{3}{4}\%$	1%
51	45.859 834	44.918 195	44.003 179	42.249 575	39.798 136
52	46.665 394	45.689 747	44.742 183	42.927 618	40.394 194
53	47.467 613	46.457 459	45.476 901	43.600 614	40.984 351
54	48.266 502	47.221 353	46.207 358	44.268 599	41.568 664
55	49.062 077	47.981 445	46.933 579	44.931 612	42.147 192
56	49.854 350	48.737 757	47.655 588	45.589 689	42.719 992
57	50.643 337	49.490 305	48.373 410	46.242 868	43.287 121
58	51.429 049	50.239 109	49.087 069	46.891 184	43.848 635
59	52.211 501	50.984 189	49.796 588	47.534 674	44.404 589
60	52.990 706	51.725 561	50.501 994	48.173 374	44.955 038
61	53.766 678	52.463 245	51.203 308	48.807 319	45.500 038
62	54.539 431	53.197 258	51.900 554	49.436 545	46.039 642
63	55.308 977	53.927 620	52.593 757	50.061 086	46.573 903
64	56.075 330	54.654 348	53.282 940	50.680 979	47.102 874
65	56.838 502	55.377 461	53.968 126	51.296 257	47.626 608
66	57.598 509	56.096 976	54.649 338	51.906 955	48.145 156
67	58.355 361	56.812 912	55.326 600	52.513 107	48.658 571
68	59.109 074	57.525 285	55.999 934	53.114 746	49.166 901
69	59.859 658	58.234 115	56.669 362	53.711 907	49.670 199
70	60.607 129	58.939 418	57.334 909	54.304 622	50.168 514
71	61.351 497	59.641 212	57.996 595	54.892 925	50.661 895
72	62.092 777	60.339 514	58.654 444	55.476 849	51.150 391
73	62.830 982	61.034 342	59.308 478	56.056 426	51.634 051
74	63.566 123	61.725 714	59.958 719	56.631 688	52.112 922
75	64.298 214	62.413 645	60.605 189	57.202 668	52.587 051
76	65.027 267	63.098 155	61.247 909	57.769 397	53.056 486
77	65.753 295	63 779 258	61.886 902	58.331 908	53.521 274
78	66.476 310	64.456 973	62.522 190	58.890 231	53.981 459
79	67.196 325	65.131 317	63.153 792	59.444 398	54.437 088
80	67.913 353	65.802 305	63.781 732	59.994 440	54.888 206
81	68.627 406	66.469 956	64.406 030	60.540 387	55.334 858
82	69.338 495	67.134 284	65.026 708	61.082 270	55.777 087
83	70.046 634	67.795 308	65.643 786	61.620 119	56.214 937
84	70.751 835	68.453 042	66.257 285	62.153 965	56.648 453
85	71.454 109	69.107 505	66.867 226	62.683 836	57.077 676
86	72.153 470	69.758 711	67.473 630	63.209 763	57.502 650
87	72.849 929	70.406 678	68.076 517	63.731 774	57.923 415
88	73.543 497	71.051 421	68.675 908	64.249 900	58.340 015
89	74.234 188	71.692 956	69.271 822	64.764 169	58.752 490
90	74.922 013	72.331 300	69.864 280	65.274 609	59.160 881
91	75.606 984	72.966 467	70.453 303	65.781 250	59.565 229
92	76.289 113	73.598 475	71.038 909	66.284 119	59.965 573
93	76.968 411	74.227 338	71.621 119	66.783 245	60.361 954
94	77.644 891	74.853 073	72.199 953	67.278 655	60.754 410
95	78.318 563	75.475 694	72.775 430	67.770 377	61.142 980
96	78.989 441	76.095 218	73.347 569	68.258 439	61.527 703
97	79.657 534	76.711 660	73.916 390	68.742 867	61.908 617
98	80.322 856	77.325 035	74.481 912	69.223 689	62.285 759
99	80.985 416	77.935 358	75.044 154	69.700 932	62.659 168
100	81.645 228	78.542 645	75.603 136	70.174 623	63.028 879

$$\frac{1-(1+r)^{-n}}{r}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
1	0.988 875	0.987 654	0.985 222	0.982 801	0.980 392
2	1.966 749	1.963 115	1.955 883	1.948 699	1.941 561
3	2.933 745	2.926 534	2.912 200	2.897 984	2.883 883
4	3.889 982	3.878 058	3.854 385	3.830 943	3.807 729
5	4.835 582	4.817 835	4.782 645	4.747 855	4.713 460
6	5.770 662	5.746 010	5.697 187	5.648 998	5.601 431
7	6.695 339	6.662 726	6.598 214	6.534 641	6.471 991
8	7.609 730	7.568 124	7.485 925	7.405 053	7.325 481
9	8.513 948	8.462 345	8.360 517	8.260 494	8.162 237
10	9.408 107	9.345 526	9.222 185	9.101 223	8.982 585
11	10.292 318	10.217 803	10.071 118	9.927 492	9.786 848
12	11.166 693	11.079 312	10.907 505	10.739 550	10.575 341
13	12.031 340	11.930 185	11.731 532	11.537 641	11.348 374
14	12.886 369	12.770 553	12.543 382	12.322 006	12.106 249
15	13.731 885	13.600 546	13.343 233	13.092 880	12.849 264
16	14.567 995	14.420 292	14.131 264	13.850 497	13.577 709
17	15.394 804	15.229 918	14.907 649	14.595 083	14.291 872
18	16.212 414	16.029 549	15.672 561	15.326 863	14.992 031
19	17.020 928	16.819 308	16.426 168	16.046 057	15.678 462
20	17.820 448	17.599 316	17.168 639	16.752 881	16.351 433
21	18.611 074	18.369 695	17.900 137	17.447 549	17.011 209
22	19.392 904	19.130 563	18 620 824	18.130 269	17.658 048
23	20.166 036	19.882 037	19.330 861	18.801 248	18.292 204
24	20.930 567	20.624 235	20.030 405	19.460 686	18.913 926
25	21.686 593	21.357 269	20.719 611	20.108 782	19.523 456
26	22.434 208	22.081 253	21.398 632	20.745 732	20.121 036
27	23.173 506	22.796 299	22.067 617	21.371 726	20.706 898
28	23.904 579	23.502 518	22.726 717	21.986 955	21.281 272
29	24.627 520	24.200 018	23.376 076	22.591 602	21.844 385
30	25.342 418	24.888 906	24.015 838	23.185 849	22.396 456
31	26.049 362	25.569 290	24.646 146	23.769 877	22.937 702
32	26.748 442	26.241 274	25.267 139	24.343 859	23.468 335
33	27.439 745	26.904 962	25.878 954	24.907 970	23.988 564
34	28.123 357	27.560 456	26.481 728	25.462 378	24.498 592
35	28.799 365	28.207 858	27.075 595	26.007 251	24.998 619
36	29.467 851	28.847 267	27.660 684	26.542 753	25.488 842
37	30.128 901	29.478 783	28.237 127	27.069 045	25.969 453
38	30.782 597	30.102 501	28.805 052	27.586 285	26.440 641
39	31.429 020	30.718 520	29.364 583	28.094 629	26.902 589
40	32.068 253	31.326 933	29.915 845	28.594 230	27.355 479
41	32.700 373	31.927 835	30.458 961	29.085 238	27.799 489
42	33.325 462	32.521 319	30.994 050	29.567 801	28.234 794
43	33.943 596	33.107 475	31.521 232	30.042 065	28.661 562
44	34.554 854	33.686 395	32.040 622	30.508 172	29.079 963
45	35.159 312	34.258 168	32.552 337	30.966 263	29.490 160
46	35.757 045	34.822 882	33.056 490	31.416 474	29.892 314
47	36.348 129	35.380 624	33.553 192	31.858 943	30.286 582
48	36.932 637	35.931 481	34.042 554	32.293 801	30.673 120
49	37.510 642	36.475 537	34.524 683	32.721 181	31.052 078
50	38.082 217	37.012 876	34.999 688	33.141 209	31.423 606

PRESENT VALUE OF ANNUITY

$$\frac{1 - (1 + r)^{-n}}{r}$$

n	$1\frac{1}{8}\%$	$1\frac{1}{4}\%$	$1\frac{1}{2}\%$	$1\frac{3}{4}\%$	2%
51	38.647 433	37.543 581	35.467 673	33.554 014	31.787 849
52	39.206 362	38.067 734	35.928 742	33.959 719	32.144 950
53	39.759 072	38.585 417	36.382 997	34.358 446	32.495 049
54	40.305 634	39.096 708	36.830 539	34.750 316	32.838 283
55	40.846 115	39.601 687	37.271 467	35.135 445	33.174 788
56	41.380 584	40.100 431	37.705 879	35.513 951	33.504 694
57	41.909 106	40.593 019	38.133 871	35.885 947	33.828 131
58	42.431 749	41.079 524	38.555 538	36.251 545	34.145 226
59	42.948 577	41.560 024	38.970 973	36.610 855	34.456 104
60	43.459 656	42.034 592	39.380 269	36.963 986	34.760 887
61	43.965 050	42.503 301	39.783 516	37.311 042	35.059 693
62	44.464 820	42.966 223	40.180 804	37.652 130	35.352 640
63	44.959 031	43.423 430	40.572 221	37.987 351	35.639 843
64	45.447 744	43.874 992	40 957 853	38.316 807	35.921 415
65	45.931 020	44.320 980	41.337 786	38.640 597	36.197 466
66	46.408 920	44.761 462	41.712 105	38.958 817	36.468 103
67	46.881 503	45.196 506	42.080 891	39.271 565	36.733 435
68	47.348 829	45.626 178	42.444 228	39.578 934	36.993 564
69	47.810 955	46.050 547	42.802 195	39.881 016	37.248 592
70	48.267 941	46.469 676	43.154 872	40.177 903	37.498 619
71	48.719 843	46.883 630	43.502 337	40.469 683	37.743 744
72	49.166 717	47.292 474	43.844 667	40.756 445	37.984 063
73	49.608 620	47.696 271	44.181 938	41.038 276	38.219 670
74	50.045 607	48.095 082	44.514 224	41.315 259	38.450 657
75	50.477 733	48.488 970	44.841 600	41.587 478	38.677 114
76	50.905 051	48.877 995	45.164 138	41.855 015	38.899 132
77	51.327 615	49.262 218	45.481 910	42.117 951	39.116 796
78	51.745 478	49.641 696	45.794 985	42.376 364	39.330 192
79	52.158 693	50.016 490	46.103 433	42.630 334	39.539 404
80	52.567 311	50.386 657	46.407 323	42.879 935	39.744 514
81	52.971 383	50.752 254	46.706 723	43.125 243	39.945 602
82	53.370 960	51.113 337	47.001 697	43.366 332	40.142 747
83	53.766 091	51.469 963	47.292 313	43.603 275	40.336 026
84	54.156 827	51.822 185	47.578 633	43.836 142	40.525 516
85	54.543 216	52.170 060	47.860 722	44.065 005	40.711 290
86	54.925 306	52.513 639	48.138 643	44.289 931	40.893 422
87	55.303 145	52.852 977	48.412 456	44.510 989	41.071 982
88	55.676 782	53.188 125	48.682 222	44.728 244	41.247 041
89	56.046 261	53.519 136	48.948 002	44.941 764	41.418 668
90	56.411 630	53.846 060	49.209 855	45.151 610	41.586 929
91	56.772 935	54.168 948	49.467 837	45.357 848	41.751 891
92	57.130 220	54.487 850	49.722 007	45.560 539	41.913 619
93	57.483 530	54.802 815	49.972 421	45.759 743	42.072 175
94	57.832 910	55.113 892	50.219 134	45.955 521	42.227 623
95	58.178 403	55.421 127	50.462 201	46.147 933	42.380 023
96	58.520 052	55.724 570	50.701 675	46.337 035	42.529 434
97	58.857 901	56.024 267	50.937 611	46.522 884	42.675 916
98	59.191 991	56.320 264	51.170 060	46.705 537	42.819 525
99	59.522 364	56.612 606	51.399 074	46.885 049	42.960 319
100	59.849 063	56.901 339	51.624 704	47.061 473	43.098 352

$$\frac{1-(1+r)^{-n}}{r}$$

n	3%	4%	5%	6%	7%
1	0.970 874	0.961 538	0.952 381	0.943 396	0.934 579
2	1.913 470	1.886 095	1.859 410	1.833 393	1.808 018
3	2.828 611	2.775 091	2.723 248	2.673 012	2.624 316
4	3.717 098	3.629 895	3.545 951	3.465 106	3.387 211
5	4.579 707	4.451 822	4.329 477	4.212 364	4.100 197
6	5.417 191	5.242 137	5.075 692	4.917 324	4.766 540
7	6.230 283	6.002 055	5.786 373	5.582 381	5.389 289
8	7.019 692	6.732 745	6.463 213	6.209 794	5.971 299
9	7.786 109	7.435 332	7.107 822	6.801 692	6.515 232
10	8.530 203	8.110 896	7.721 735	7.360 087	7.023 582
11	9.252 624	8.760 477	8.306 414	7.886 875	7.498 674
12	9.954 004	9.385 074	8.863 252	8.383 844	7.942 686
13	10.634 955	9.985 648	9.393 573	8.852 683	8.357 651
14	11.296 073	10.563 123	9.898 641	9.294 984	8.745 468
15	11.937 935	11.118 387	10.379 658	9.712 249	9.107 914
16	12.561 102	11.652 296	10.837 770	10.105 895	9.446 649
17	13.166 118	12.165 669	11.274 066	10.477 260	9.763 223
18	13.753 513	12.659 297	11.689 587	10.827 603	10.059 087
19	14.323 799	13.133 939	12.085 321	11.158 116	10.335 595
20	14.877 475	13.590 326	12.462 210	11.469 921	10.594 014
21	15.415 024	14.029 160	12.821 153	11.764 077	10.835 527
22	15.936 917	14.451 115	13.163 003	12.041 582	11.061 240
23	16.443 608	14.856 842	13.488 574	12.303 379	11.272 187
24	16.935 542	15.246 963	13.798 642	12.550 358	11.469 334
25	17.413 148	15.622 080	14.093 945	12.783 356	11.653 583
26	17.876 842	15.982 769	14.375 185	13.003 166	11.825 779
27	18.327 031	16.329 586	14.643 034	13.210 534	11.986 709
28	18.764 108	16.663 063	14.898 127	13.406 164	12.137 111
29	19.188 455	16.983 715	15.141 074	13.590 721	12.277 674
30	19.600 441	17.292 033	15.372 451	13.764 831	12.409 041
31	20.000 428	17.588 494	15.592 811	13.929 086	12.531 814
32	20.388 766	17.873 551	15.802 677	14.084 043	12.646 555
33	20.765 792	18.147 646	16.002 549	14.230 230	12.753 790
34	21.131 837	18.411 198	16.192 904	14.368 141	12.854 009
35	21.487 220	18.664 613	16.374 194	14.498 246	12.947 672
36	21.832 252	18.908 282	16.546 852	14.620 987	13.035 208
37	22.167 235	19.142 579	16.711 287	14.736 780	13.117 017
38	22.492 462	19.367 864	16.867 893	14.846 019	13.193 473
39	22.808 215	19.584 485	17.017 041	14.949 075	13.264 928
40	23.114 772	19.792 774	17.159 086	15.046 297	13.331 709
41	23.412 400	19.993 052	17.294 368	15.138 016	13.394 120
42	23.701 359	20.185 627	17.423 208	15.224 543	13.452 449
43	23.981 902	20.370 795	17.545 912	15.306 173	13.506 962
44	24.254 274	20.548 841	17.662 773	15.383 182	13.557 908
45	24.518 713	20.720 040	17.774 070	15.455 832	13.605 522
46	24.775 449	20.884 654	17.880 066	15.524 370	13.650 020
47	25.024 708	21.042 936	17.981 016	15.589 028	13.691 608
48	25.266 707	21.195 131	18.077 158	15.650 027	13.730 474
49	25.501 657	21.341 472	18.168 722	15.707 572	13.766 799
50	25.729 764	21.482 185	18.255 925	15.761 861	13.800 746

$$\frac{1-(1+r)^{-n}}{r}$$

n	3%	4%	5%	6%	7%
51	25.951 227	21.617 485	18.338 977	15.813 076	13.832 473
52	26.166 240	21.747 582	18.418 073	15.861 393	13.862 124
53	26.374 990	21.872 675	18.493 403	15.906 974	13.889 836
54	26.577 660	21.992 957	18.565 146	15.949 976	13.915 735
55	26.774 428	22.108 612	18.633 472	15.990 543	13.939 939
56	26.965 464	22.219 819	18.698 545	16.028 814	13.962 560
57	27.150 936	22.326 749	18.760 519	16.064 919	13.983 701
58	27.331 005	22.429 567	18.819 542	16.098 980	14.003 458
59	27.505 831	22.528 430	18.875 754	16.131 113	14.021 924
60	27.675 564	22.623 490	18.929 290	16.161 428	14.039 181
61	27.840 353	22.714 894	18.980 276	16.190 026	14.055 309
62	28.000 343	22.802 783	19.028 834	16.217 006	14.070 383
63	28.155 673	22.887 291	19.075 080	16.242 458	14.084 470
64	28.306 478	22.968 549	19.119 124	16.266 470	14.097 635
65	28.452 892	23.046 682	19.161 070	16.289 123	14.109 940
66	28.595 040	23.121 810	19.201 019	16.310 493	14.121 439
67	28.733 049	23.194 048	19.239 066	16.330 654	14.132 186
68	28.867 038	23.263 507	19.275 301	16.349 673	14.142 230
69	28.997 124	23.330 296	19.309 810	16.367 617	14.151 617
70	29.123 421	23.394 515	19.342 677	16.384 544	14.160 389
71	29.246 040	23.456 264	19.373 978	16.400 513	14.168 588
72	29.365 088	23.515 639	19.403 788	16.415 578	14.176 251
73	29.480 667	23.572 730	19.432 179	16.429 791	14.183 412
74	29.592 881	23.627 625	19.459 218	16.443 199	14.190 104
75	29.701 826	23.680 408	19.484 970	16.455 848	14.196 359
76	29.807 598	23.731 162	19.509 495	16.467 781	14.202 205
77	29.910 290	23.779 963	19.532 853	16.479 039	14.207 668
78	30.009 990	23.826 888	19.555 098	16.489 659	14.212 774
79	30.106 786	23.872 008	19.576 284	16.499 679	14.217 546
80	30.200 763	23.915 392	19.596 460	16.509 131	14.222 005
81	30.292 003	23.957 108	19.615 677	16.518 048	14.226 173
82	30.380 586	23.997 219	19.633 978	16.526 460	14.230 069
83	30.466 588	24.035 787	19.651 407	16.534 396	14.233 709
84	30.550 086	24.072 872	19.668 007	16.541 883	14.237 111
85	30.631 151	24.108 531	19.683 816	16.548 947	14.240 291
86	30.709 855	24.142 818	19.698 873	16.555 610	14.243 262
87	30.786 267	24.175 787	19.713 212	16.561 896	14.246 040
88	30.860 454	24.207 487	19.726 869	16.567 827	14.248 635
89	30.932 479	24.237 969	19.739 875	16.573 421	14.251 061
90	31.002 407	24.267 278	19.752 262	16.578 699	14.253 328
91	31.070 298	24.295 459	19.764 059	16.583 679	14.255 447
92	31.136 212	24.322 557	19.775 294	16.588 376	14.257 427
93	31.200 206	24.348 612	19.785 994	16.592 808	14.259 277
94	31.262 336	24.373 666	19.796 185	16.596 988	14.261 007
95	31.322 656	24.397 756	19.805 891	16.600 932	14.262 623
96	31.381 219	24.420 919	19.815 134	16.604 653	14.264 134
97	31.438 077	24.443 191	19.823 937	16.608 163	14.265 546
98	31.493 279	24.464 607	19.832 321	16.611 475	14.266 865
99	31.546 872	24.485 199	19.840 306	16.614 599	14.268 098
100	31.598 905	24.504 999	19.847 910	16.617 546	14.269 251

Table VII

TABLE OF MORTALITY

Note: This table has been adapted from the 1941 Commissioners Standard Ordinary (CSO) Mortality Table. In the form given, the table may be thought of as a probability distribution of the number of deaths according to age. For example, in the fourth column we read 0.00243 as the probability of dying between ages 20 and 21. The sum of the probabilities in this column through 0.00243, or 0.07467, is the probability of dying before reaching the age of 21 years.

MORTALITY TABLE
(Adapted from 1941 CSO Table)

Age x	Number living l_x	No. of deaths d_x	Yearly probability of dying q_x	Age x	Number living l_x	No. of deaths d_x	Yearly probability of dying q_x
0	1,023,102	23,102	0.02258	**50**	810,900	9,990	0.01232
1	1,000,000	5,770	0.00577	51	800,910	10,628	0.01327
2	994,230	4,116	0.00414	52	790,282	11,301	0.01430
3	990,114	3,347	0.00338	53	778,981	12,020	0.01543
4	986,767	2,950	0.00299	54	766,961	12,770	0.01665
5	983,817	2,715	0.00276	55	754,191	13,560	0.01798
6	981,102	2,561	0.00261	56	740,631	14,390	0.01943
7	978,541	2,417	0.00247	57	726,241	15,251	0.02100
8	976,124	2,255	0.00231	58	710,990	16,147	0.02271
9	973,869	2,065	0.00212	59	694,843	17,072	0.02457
10	971,804	1,914	0.00197	**60**	677,771	18,022	0.02659
11	969,890	1,852	0.00191	61	659,749	18,988	0.02878
12	968,038	1,859	0.00192	62	640,761	19,979	0.03118
13	966,179	1,913	0.00198	63	620,782	20,958	0.03376
14	964,266	1,996	0.00207	64	599,824	21,942	0.03658
15	962,270	2,069	0.00215	65	577,882	22,907	0.03964
16	960,201	2,103	0.00219	66	554,975	23,842	0.04296
17	958,098	2,156	0.00225	67	531,133	24,730	0.04656
18	955,942	2,199	0.00230	68	506,403	25,553	0.05046
19	953,743	2,260	0.00237	69	480,850	26,302	0.05470
20	951,483	2,312	0.00243	**70**	454,548	26,955	0.05930
21	949,171	2,382	0.00251	71	427,593	27,481	0.06427
22	946,789	2,452	0.00259	72	400,112	27,872	0.06966
23	944,337	2,531	0.00268	73	372,240	28,104	0.07550
24	941,806	2,609	0.00277	74	344,136	28,154	0.08181
25	939,197	2,705	0.00288	75	315,982	28,009	0.08864
26	936,492	2,800	0.00299	76	287,973	27,651	0.09602
27	933,692	2,904	0.00311	77	260,322	27,071	0.10399
28	930,788	3,025	0.00325	78	233,251	26,262	0.11259
29	927,763	3,154	0.00340	79	206,989	25,224	0.12186
30	924,609	3,292	0.00356	**80**	181,765	23,966	0.13185
31	921,317	3,437	0.00373	81	157,799	22,502	0.14260
32	917,880	3,598	0.00392	82	135,297	20,857	0.15416
33	914,282	3,767	0.00412	83	114,440	19,062	0.16657
34	910,515	3,961	0.00435	84	95,378	17,157	0.17988
35	906,554	4,161	0.00459	85	78,221	15,185	0.19413
36	902,393	4,386	0.00486	86	63,036	13,198	0.20937
37	898,007	4,625	0.00515	87	49,838	11,245	0.22563
38	893,382	4,878	0.00546	88	38,593	9,378	0.24300
39	888,504	5,162	0.00581	89	29,215	7,638	0.26144
40	883,342	5,459	0.00618	**90**	21,577	6,063	0.28099
41	877,883	5,785	0.00659	91	15,514	4,681	0.30173
42	872,098	6,131	0.00703	92	10,833	3,506	0.32364
43	865,967	6,503	0.00751	93	7,327	2,540	0.34666
44	859,464	6,910	0.00804	94	4,787	1,776	0.37100
45	852,554	7,340	0.00861	95	3,011	1,193	0.39621
46	845,214	7,801	0.00923	96	1,818	813	0.44719
47	837,413	8,299	0.00991	97	1,005	551	0.54826
48	829,114	8,822	0.01064	98	454	329	0.72467
49	820,292	9,392	0.01145	99	125	125	1.00000

Table VIII

TABLE OF THE NORMAL CURVE

Note: The table gives the percentage of the area under the normal curve from the ordinate at $t = 0$ to the ordinate for the value of t given in the column at the left. Values of the ordinate are given in the third column. Because the normal curve is symmetrical with respect to the ordinate at $t = 0$, tabular values are the same for positive and negative t.

t	Area	Ordinate	t	Area	Ordinate	t	Area	Ordinate	t	Area	Ordinate
.00	.0000	.3989	**.50**	.1915	.3521	**1.00**	.3413	.2420	**1.50**	.4332	.1295
.01	.0040	.3989	.51	.1950	.3503	1.01	.3438	.2396	1.51	.4345	.1276
.02	.0080	.3989	.52	.1985	.3485	1.02	.3461	.2371	1.52	.4357	.1257
.03	.0120	.3988	.53	.2019	.3467	1.03	.3485	.2347	1.53	.4370	.1238
.04	.0160	.3986	.54	.2054	.3448	1.04	.3508	.2323	1.54	.4382	.1219
.05	.0199	.3984	.55	.2088	.3429	1.05	.3531	.2299	1.55	.4394	.1200
.06	.0239	.3982	.56	.2123	.3411	1.06	.3554	.2275	1.56	.4406	.1182
.07	.0279	.3980	.57	.2157	.3391	1.07	.3577	.2251	1.57	.4418	.1163
.08	.0319	.3977	.58	.2190	.3372	1.08	.3599	.2227	1.58	.4430	.1145
.09	.0359	.3973	.59	.2224	.3352	1.09	.3621	.2203	1.59	.4441	.1127
.10	.0398	.3970	**.60**	.2258	.3332	**1.10**	.3643	.2179	**1.60**	.4452	.1109
.11	.0438	.3965	.61	.2291	.3312	1.11	.3665	.2155	1.61	.4463	.1092
.12	.0478	.3961	.62	.2324	.3292	1.12	.3686	.2131	1.62	.4474	.1074
.13	.0517	.3956	.63	.2357	.3271	1.13	.3708	.2107	1.63	.4485	.1057
.14	.0557	.3951	.64	.2389	.3251	1.14	.3729	.2083	1.64	.4495	.1040
.15	.0596	.3945	.65	.2422	.3230	1.15	.3749	.2059	1.65	.4505	.1023
.16	.0636	.3939	.66	.2454	.3209	1.16	.3770	.2036	1.66	.4515	.1006
.17	.0675	.3932	.67	.2486	.3187	1.17	.3790	.2012	1.67	.4525	.0989
.18	.0714	.3925	.68	.2518	.3166	1.18	.3810	.1989	1.68	.4535	.0973
.19	.0754	.3918	.69	.2549	.3144	1.19	.3830	.1965	1.69	.4545	.0957
.20	.0793	.3910	**.70**	.2580	.3123	**1.20**	.3849	.1942	**1.70**	.4554	.0941
.21	.0832	.3902	.71	.2612	.3101	1.21	.3869	.1919	1.71	.4564	.0925
.22	.0871	.3894	.72	.2642	.3079	1.22	.3888	.1895	1.72	.4573	.0909
.23	.0910	.3885	.73	.2673	.3056	1.23	.3907	.1872	1.73	.4582	.0893
.24	.0948	.3876	.74	.2704	.3034	1.24	.3925	.1849	1.74	.4591	.0878
.25	.0987	.3867	.75	.2734	.3011	1.25	.3944	.1827	1.75	.4599	.0863
.26	.1026	.3857	.76	.2764	.2989	1.26	.3962	.1804	1.76	.4608	.0848
.27	.1064	.3847	.77	.2794	.2966	1.27	.3980	.1781	1.77	.4616	.0833
.28	.1103	.3836	.78	.2823	.2943	1.28	.3997	.1759	1.78	.4625	.0818
.29	.1141	.3825	.79	.2852	.2920	1.29	.4015	.1736	1.79	.4633	.0804
.30	.1179	.3814	**.80**	.2881	.2897	**1.30**	.4032	.1714	**1.80**	.4641	.0790
.31	.1217	.3802	.81	.2910	.2874	1.31	.4049	.1692	1.81	.4649	.0775
.32	.1255	.3790	.82	.2939	.2850	1.32	.4066	.1669	1.82	.4656	.0761
.33	.1293	.3778	.83	.2967	.2827	1.33	.4082	.1647	1.83	.4664	.0748
.34	.1331	.3765	.84	.2996	.2803	1.34	.4099	.1626	1.84	.4671	.0734
.35	.1368	.3752	.85	.3023	.2780	1.35	.4115	.1604	1.85	.4678	.0721
.36	.1406	.3739	.86	.3051	.2756	1.36	.4131	.1582	1.86	.4686	.0707
.37	.1443	.3726	.87	.3079	.2732	1.37	.4147	.1561	1.87	.4693	.0694
.38	.1480	.3712	.88	.3106	.2709	1.38	.4162	.1540	1.88	.4700	.0681
.39	.1517	.3697	.89	.3133	.2685	1.39	.4177	.1518	1.89	.4706	.0669
.40	.1554	.3683	**.90**	.3159	.2661	**1.40**	.4192	.1497	**1.90**	.4713	.0656
.41	.1591	.3668	.91	.3186	.2637	1.41	.4207	.1476	1.91	.4719	.0644
.42	.1628	.3653	.92	.3212	.2613	1.42	.4222	.1456	1.92	.4726	.0632
.43	.1664	.3637	.93	.3238	.2589	1.43	.4236	.1435	1.93	.4732	.0620
.44	.1700	.3621	.94	.3264	.2565	1.44	.4251	.1415	1.94	.4738	.0608
.45	.1736	.3605	.95	.3289	.2541	1.45	.4265	.1394	1.95	.4744	.0596
.46	.1772	.3589	.96	.3315	.2516	1.46	.4279	.1374	1.96	.4750	.0584
.47	.1808	.3572	.97	.3340	.2492	1.47	.4292	.1354	1.97	.4756	.0573
.48	.1844	.3555	.98	.3365	.2468	1.48	.4306	.1334	1.98	.4762	.0562
.49	.1879	.3538	.99	.3389	.2444	1.49	.4319	.1315	1.99	.4767	.0551
.50	.1915	.3521	**1.00**	.3413	.2420	**1.50**	.4332	.1295	**2.00**	.4773	.0540

t	Area	Ordinate	t	Area	Ordinate	t	Area	Ordinate	t	Area	Ordinate
2.00	.4773	.0540	**2.50**	.4938	.0175	**3.00**	.4987	.0044	**3.50**	.4998	.0009
2.01	.4778	.0529	2.51	.4940	.0171	3.01	.4987	.0043	3.51	.4998	.0008
2.02	.4783	.0519	2.52	.4941	.0167	3.02	.4987	.0042	3.52	.4998	.0008
2.03	.4788	.0508	2.53	.4943	.0163	3.03	.4988	.0041	3.53	.4998	.0008
2.04	.4793	.0498	2.54	.4945	.0159	3.04	.4988	.0039	3.54	.4998	.0008
2.05	.4798	.0488	2.55	.4946	.0155	3.05	.4989	.0038	3.55	.4998	.0007
2.06	.4803	.0478	2.56	.4948	.0151	3.06	.4989	.0037	3.56	.4998	.0007
2.07	.4808	.0468	2.57	.4949	.0147	3.07	.4989	.0036	3.57	.4998	.0007
2.08	.4812	.0459	2.58	.4951	.0143	3.08	.4990	.0035	3.58	.4998	.0007
2.09	.4817	.0449	2.59	.4952	.0139	3.09	.4990	.0034	3.59	.4998	.0006
2.10	.4821	.0440	**2.60**	.4953	.0136	**3.10**	.4990	.0033	**3.60**	.4998	.0006
2.11	.4826	.0431	2.61	.4955	.0132	3.11	.4991	.0032	3.61	.4999	.0006
2.12	.4830	.0422	2.62	.4956	.0129	3.12	.4991	.0031	3.62	.4999	.0006
2.13	.4834	.0413	2.63	.4957	.0126	3.13	.4991	.0030	3.63	.4999	.0006
2.14	.4838	.0404	2.64	.4959	.0122	3.14	.4992	.0029	3.64	.4999	.0005
2.15	.4842	.0396	2.65	.4960	.0119	3.15	.4992	.0028	3.65	.4999	.0005
2.16	.4846	.0387	2.66	.4961	.0116	3.16	.4992	.0027	3.66	.4999	.0005
2.17	.4850	.0379	2.67	.4962	.0113	3.17	.4992	.0026	3.67	.4999	.0005
2.18	.4854	.0371	2.68	.4963	.0110	3.18	.4993	.0025	3.68	.4999	.0005
2.19	.4857	.0363	2.69	.4964	.0107	3.19	.4993	.0025	3.69	.4999	.0004
2.20	.4861	.0355	**2.70**	.4965	.0104	**3.20**	.4993	.0024	**3.70**	.4999	.0004
2.21	.4865	.0347	2.71	.4966	.0101	3.21	.4993	.0023	3.71	.4999	.0004
2.22	.4868	.0339	2.72	.4967	.0099	3.22	.4994	.0022	3.72	.4999	.0004
2.23	.4871	.0332	2.73	.4968	.0096	3.23	.4994	.0022	3.73	.4999	.0004
2.24	.4875	.0325	2.74	.4969	.0094	3.24	.4994	.0021	3.74	.4999	.0004
2.25	.4878	.0317	2.75	.4970	.0091	3.25	.4994	.0020	3.75	.4999	.0004
2.26	.4881	.0310	2.76	.4971	.0089	3.26	.4994	.0020	3.76	.4999	.0003
2.27	.4884	.0303	2.77	.4972	.0086	3.27	.4995	.0019	3.77	.4999	.0003
2.28	.4887	.0297	2.78	.4973	.0084	3.28	.4995	.0018	3.78	.4999	.0003
2.29	.4890	.0290	2.79	.4974	.0081	3.29	.4995	.0018	3.79	.4999	.0003
2.30	.4893	.0283	**2.80**	.4974	.0079	**3.30**	.4995	.0017	**3.80**	.4999	.0003
2.31	.4896	.0277	2.81	.4975	.0077	3.31	.4995	.0017	3.81	.4999	.0003
2.32	.4898	.0271	2.82	.4976	.0075	3.32	.4996	.0016	3.82	.4999	.0003
2.33	.4901	.0264	2.83	.4977	.0073	3.33	.4996	.0016	3.83	.4999	.0003
2.34	.4904	.0258	2.84	.4977	.0071	3.34	.4996	.0015	3.84	.4999	.0003
2.35	.4906	.0252	2.85	.4978	.0069	3.35	.4996	.0015	3.85	.4999	.0002
2.36	.4909	.0246	2.86	.4979	.0067	3.36	.4996	.0014	3.86	.4999	.0002
2.37	.4911	.0241	2.87	.4980	.0065	3.37	.4996	.0014	3.87	.5000	.0002
2.38	.4913	.0235	2.88	.4980	.0063	3.38	.4996	.0013	3.88	.5000	.0002
2.39	.4916	.0229	2.89	.4981	.0061	3.39	.4997	.0013	3.89	.5000	.0002
2.40	.4918	.0224	**2.90**	.4981	.0060	**3.40**	.4997	.0012	**3.90**	.5000	.0002
2.41	.4920	.0219	2.91	.4982	.0058	3.41	.4997	.0012	3.91	.5000	.0002
2.42	.4922	.0213	2.92	.4983	.0056	3.42	.4997	.0012	3.92	.5000	.0002
2.43	.4925	.0208	2.93	.4983	.0055	3.43	.4997	.0011	3.93	.5000	.0002
2.44	.4927	.0203	2.94	.4984	.0053	3.44	.4997	.0011	3.94	.5000	.0002
2.45	.4929	.0198	2.95	.4984	.0051	3.45	.4997	.0010	3.95	.5000	.0002
2.46	.4931	.0194	2.96	.4985	.0050	3.46	.4997	.0010	3.96	.5000	.0002
2.47	.4932	.0189	2.97	.4985	.0049	3.47	.4997	.0010	3.97	.5000	.0002
2.48	.4934	.0184	2.98	.4986	.0047	3.48	.4998	.0009	3.98	.5000	.0001
2.49	.4936	.0180	2.99	.4986	.0046	3.49	.4998	.0009	3.99	.5000	.0001
2.50	.4938	.0175	**3.00**	.4987	.0044	**3.50**	.4998	.0009	**4.00**	.5000	.0001

Answers

Note: Answers are given for the odd-numbered problems having only one part and for alternate parts of all problems, odd and even, having more than one part. In a few instances, no answers are given, and in some, all answers are given, depending upon the nature of the problems and the instructional intent.

Section 1. Page 2

Problems:

1. (*a*) 316.98 (*c*) 3,342 (*e*) 460,846,197.19

4. (*a*) 3,260 (*c*) 2,842.128

5. Total sales, $3,824.23

6. (*a*) 1,536 (*b*) 1,649 (*c*) 1,719 (*d*) 10,088 (*e*) 14,228 (*f*) 25,816 (*g*) 7,142 (*h*) 18,681 (*i*) 12,915 (*j*) 11,042 (*k*) 7,464.92 (*l*) 18,907.6 (*m*) 3,407.43 (*n*) 17.37 (*o*) 2,797.63

Section 2. Page 5

Problems:

1. (*a*) 267 (*b*) 32,100 (*c*) 4,515.81 (*d*) 1,180,927 (*e*) 41.91

2. (*a*) 35 (*b*) 12 (*c*) 38 (*d*) 44 (*e*) 235 (*f*) 198 (*g*) 2,686 (*h*) 255,908 (*i*) 58.74 (*j*) 1.73 (*k*) 244.29 (*l*) 87.69

3. (*a*) $1.25 (*b*) $2.70 (*c*) $3.35 (*d*) $3.55 (*e*) $0.13 (*f*) $1.84 (*g*) $0.28 (*h*) $2.72 (*i*) $2.68 (*j*) $4.87 (*k*) $1.09 (*l*) $3.42

4. (*a*) $7.41 (*b*) $5.83 (*c*) $3.34 (*d*) $1.77 (*e*) $6.22 (*f*) $4.18 (*g*) $2.65 (*h*) $0.56 (*i*) $8.73 (*j*) $7.67 (*k*) $6.52 (*l*) $5.44

5. Total net profit, $487.38

7. (*a*) 214,244 (*b*) 307,603 (*c*) 149,469 (*d*) 318.07 (*e*) 156,500,106 (*f*) 1,200,464 (*g*) 316,210 (*h*) 157,740 (*i*) 256,401 (*j*) 3,278,790 (*k*) 291,745.03 (*l*) 27,579.79 (*m*) 44,727 (*n*) 1,563.499 (*o*) 420.4829

Section 3. Page 8

Problems:

1. (*a*) 6,000 (*b*) 35,244 (*c*) $1,192 (*d*) 1,515 (*e*) 1,176 (*f*) 600 (*g*) 3,264.9 (*h*) 75,235 (*i*) $850 (*j*) 1,280 (*k*) 3,620 (*l*) 24,500 (*m*) 5,790 (*n*) 675 (*o*) 25,500 (*p*) 9,300 (*q*) 21,420 (*r*) 163,650 (*s*) 10,692 (*t*) 69,894 (*u*) 960,454 (*v*) 997,931 (*w*) 17,753,582

2. (*a*) 768 (*b*) 1,325 (*c*) 6,764 (*d*) 1,755 (*e*) 5,808 (*f*) 2,793

3. Total pay roll, $40,857.41

4. (*a*) 11,302,936 (*b*) 412,137 (*c*) 10,970,624 (*d*) 608,124 (*e*) 10,477,120 (*f*) 6,329,112 (*g*) 3,317,072 (*h*) 30,765,795 (*i*) 4,198,968 (*j*) 62,916

5. 616; 2,387; 5,379; 38.06; 11.66; 29,403

6. By 25: 1,100; $11,250; $115; 25.875; $3,755; 0.175
By 50: 2,200; $22,500; $230; 51.75; $7,510; 0.35
By 99: 4,356; $44,550; $455.40; 102.465; $14,869.80; 0.693
By 101: 4,444; $45,450; $464.60; 104.535; $15,170.20; 0.707

Section 4. Page 9

Problems:

1. (*a*) 259, remainder, 296 (*c*) 196, remainder, 310 (*e*) 75, remainder, 595

3. (*a*) 279, remainder, 13 (*b*) 6,660, remainder, 114 (*c*) 1,089, remainder, 34 (*d*) 1,043, remainder, 44 (*e*) 961, remainder, 18 (*f*) 798, remainder, 12 (*g*) 1,000, remainder, 356 (*h*) 1,200, remainder, 168 (*i*) 427, remainder, 39 (*j*) 2,338, remainder, 40

Section 5. Page 11

Problems:

1. (*a*) 2, 2, 2, 3, 7 (*c*) 3, 11, 11 (*e*) 2, 2, 2, 2, 7, 7 (*g*) 5, 5, 5, 3, 3 (*i*) 2, 19, 19

3. 7, 5; 3, 7; 7, 11; 11, 11; 7, 7
5. 5
7. 5, 7, 11, 13
9. 6,075, 9,636, 8,124, 1,872
11. 5,004, 9,009, 6,264

Section 6. Page 13

Problems:

1. (*a*) 18 (*c*) 8 (*e*) 4 **3.** 25 **5.** 3

Section 7. Page 15

Problems:

1. (*a*) 216 (*c*) 7,260 (*e*) 11,220
2. (*a*) 90 (*c*) 1,134 (*e*) 2,520
3. 50 = LCM of 5, 10, and 25

Section 9. Page 17

Problems:

1. (*a*) $1\frac{1}{4}$ (*c*) $12\frac{37}{42}$
2. (*a*) $\frac{2}{3}$ (*c*) $\frac{2}{5}$ (*e*) $\frac{41}{61}$ (*g*) $\frac{7}{18}$
3. (*a*) $\frac{1}{5}$ (*c*) $1\frac{1}{5}$ (*e*) $\frac{2}{3}$ (*g*) $32\frac{1}{2}$

4. (*a*) 2 (*c*) $2\frac{3}{16}$ (*e*) $\frac{17}{18}$ (*g*) $5\frac{2}{3}$ (*i*) $40\frac{4}{15}$
5. (*a*) $\frac{1}{3}$ (*c*) $\frac{5}{12}$ (*e*) $\frac{1}{36}$ (*g*) $8\frac{1}{2}$ (*i*) $64\frac{4}{39}$
7. 27 min, 3 min, 1 hr, 19 min
9. \$154$\frac{5}{8}$, \$100$\frac{6}{8}$, \$57$\frac{1}{8}$

Section 10. Page 19

Problems:

1. (*a*) $\frac{2}{3}$ (*b*) $\frac{3}{16}$ (*c*) $\frac{3}{4}$ (*d*) $\frac{3}{5}$ (*e*) $\frac{22}{35}$ (*f*) $\frac{9}{35}$ (*g*) $\frac{5}{8}$ (*h*) $\frac{3}{20}$ (*i*) $\frac{2}{9}$ (*j*) $\frac{5}{6}$ (*k*) $\frac{3}{10}$ (*l*) $\frac{1}{2}$ (*m*) $\frac{3}{8}$ (*n*) $\frac{2}{5}$ (*o*) 5 (*p*) 12
2. (*a*) $3\frac{1}{5}$ (*c*) 96
3. (*a*) $8\frac{8}{9}$ (*c*) $208\frac{1}{100}$ (*e*) $29\frac{1}{6}$ (*g*) $134\frac{3}{4}$ (*i*) $264\frac{3}{8}$ (*k*) 6 (*m*) 18 (*o*) $\frac{1}{506}$
5. \$43.75, \$156.25, \$106.25

Section 11. Page 21

Problems:

1. (*a*) 2 (*c*) $\frac{8}{15}$ (*e*) 48 (*g*) $\frac{21}{32}$ (*i*) $2\frac{80}{203}$
2. (*a*) $\frac{1}{8}$ (*c*) $1\frac{1}{8}$ (*e*) $1\frac{1}{3}$ (*g*) 1 (*i*) 20 (*k*) 1 (*m*) $4\frac{10}{21}$ (*o*) $1\frac{1}{2}$ (*q*) $20\frac{32}{71}$
3. (*a*) $\frac{16}{31}$ (*c*) $1\frac{17}{100}$ (*e*) $1\frac{1}{11}$ (*g*) $\frac{1}{18}$ (*i*) $\frac{827}{5220}$ (*k*) 2

5. $\frac{4}{9}$ 7. $\frac{7}{9}$ 9. 16¢

Section 12. Page 23

Problems:

1. (*a*) Twenty-five hundredths (*c*) two thousandths
(*e*) two hundred three thousandths (*g*) ninety-two ten-thousandths
(*i*) two hundred forty three thousand six hundred seventy-five millionths
(*k*) two hundred ninety-five and sixty-five hundredths
(*m*) one thousand one and one thousand one ten-thousandths
(*o*) seven hundred and seven hundredths
2. (*a*) $\frac{1}{4}$ (*c*) $\frac{59}{200}$ (*e*) $\frac{3}{8}$ (*g*) $\frac{1}{8}$ (*i*) $\frac{1}{7}$ (*k*) $\frac{1}{3}$

Section 13. Page 23

Problems:

1. 387.596
2. (*a*) 71.875 (*c*) \$26.38
3. Net profit, \$29,373.66
5. Balance, \$465.45

Section 14. Page 25

Problems:

1. (*a*) 0.009 (*c*) 0.48 (*e*) 27.95 (*g*) 37.50 (*i*) 64.6875
2. (*a*) \$1.95 (*c*) \$7.04 (*e*) \$3.91 (*g*) \$0.83
3. (*a*) \$183 (*b*) \$606.66
5. Total cost, \$9,809.25

Section 15. Page 28

Problems:

1. (*a*) 25.08 (*c*) 53.533 (*e*) 0.003
2. (*a*) 11.29 (*c*) 1.60 (*e*) 3.22 (*g*) 0.02 (*i*) 0.68295
3. \$18.75
4. (*a*) 700 (*c*) 256 (*e*) 34
5. \$5.50
7. 320 lb, 3,888$\frac{8}{9}$ lb, 416 cakes, 150 lb

Section 18. Page 34

Problems:

1. 2.67 ft, 3
2. (*a*) \$674.35 (*c*) \$5,864.04
3. 537.8803424, 537.88
4. (*a*) 16.676 (*c*) 0.051
5. 1.26
7. (*a*) 2 (*c*) 3 (*e*) 1 (*g*) 3 (*i*) 5 (*k*) 3 (*m*) 4 (*o*) 2 (*q*) 3 (*s*) 7 (*u*) 7
8. (*a*) 7.3 (*c*) 0.108 (*e*) 112,200
9. (*a*) 14 (*c*) 0.0036 (*e*) 20
10. (*a*) 166.46 (*c*) 84.0
11. (*a*) 111.41 (*c*) 0.0022

Section 19(a). Page 36

Problems:

1. 632.58, excess 6
2. (*a*) 316.98, excess 0 (*b*) 709.95, excess 3 (*c*) 3,342, excess 3 (*d*) 270,692.08, excess 7 (*e*) 460,846,197.19, excess 1

Chapter I. Page 38

Exercises:

1. (*a*) $1\frac{3}{8}$ (*c*) $\frac{5}{12}$ (*e*) $\frac{1}{7}$ (*g*) $5\frac{7}{16}$ (*i*) $\frac{3}{5}$
2. (*a*) $1\frac{3}{8}$ (*c*) $15\frac{17}{60}$ (*e*) $1\frac{1553}{2520}$
3. (*a*) $\frac{1}{8}$ (*c*) $7\frac{29}{39}$ (*e*) $2\frac{17}{18}$ (*g*) $31\frac{9}{20}$
4. (*a*) $5\frac{1}{2}$ (*c*) $26\frac{1}{4}$ (*e*) $259\frac{1}{5}$ (*g*) $1\frac{3}{25}$ (*i*) $\frac{63}{130}$ (*k*) 22
5. $\frac{27}{32}$ in.
7. \$71.28
8. (*a*) 66 ft/sec (*c*) $469\frac{1}{3}$ ft/sec
9. (*a*) $17\frac{1}{22}$ mph (*b*) 60 mph (*c*) $109\frac{1}{11}$ mph
10. (*a*) 0.017 (*c*) 0.016
11. No, $6\frac{13}{25}$ in., $5\frac{22}{25}$ in.
13. 11 pieces, 1.21 in.
15. (*a*) 86 (*c*) 11 (*e*) 22
16. (*a*) 48 (*c*) 918
17. 0.264 lb/in.
18. (*a*) \$11.25 (*c*) \$78.75
19. \$19,395
21. 64 lots
23. \$24.48
25. \$60,000
27. $\frac{2}{15}$
29. \$12.64
31. $23\frac{1}{12}$ in.
33. 24 qt
35. \$1.94
37. $\frac{7}{9}$
39. 15
41. 0.027
42. (*a*) 81 (*c*) 78
43. 53,210 lb
45. \$79,125, \$23,925

47. (*a*) 11.29°C (*b*) −3.76°C (*c*) −31.48°C
48. (*a*) 3.1429 (*c*) 6.75 (*e*) 0.18

49. 1,000 man-hr
51. \$0.1056 per sq ft
53. 1,322
55. \$4.02
57. \$70.38
59. 550 cu in.
61. \$229.75

Chapter I. Page 42

Self-Test:

1. \$17,099.40, \$3,032.35
2. 2

3. (*a*) 5 (*b*) 1 (*c*) 5 (*d*) 4, 5, 6, or 7 (*e*) 5

4. \$16,825.53; excess 3
5. 17,640, LCM
6. 18
7. 100.52
8. $\frac{1}{15}$
9. \$45.36

10. Management, \$5,882.40; advertising, \$11,223.00; sales, \$15,091.20; collections, \$2,254.00; total pay roll, \$34,450.60

Section 20. Page 48

Problems:

1. (*a*) $a + b + c$ (*c*) abc (*e*) $(a/b) - c$ (*g*) $abc + bc$
2. (*a*) Prt (*c*) $(x + y)/2$

3. $x/12$¢
5. $20y$ miles
7. $x + \frac{2}{5}x = 90$
9. $b = 2a$
11. $w - 15 = z$
13. $C = S/12$

15. $4y$, $5y$
17. y/t, $(Ty)/t$, $(4y)/t$, $(k + \frac{1}{2})(y/t)$

Section 23. Page 52

Problems:

1. −8
3. −3
5. −108
7. 136
9. −0.114
11. $1\frac{1}{2}$
13. $-12\frac{13}{24}$
15. $-10\frac{21}{40}$
17. $99\frac{27}{35}$
19. −99
21. 0
23. −1
25. −11.98
27. −0.9
29. $-9\frac{5}{12}$
31. −1
33. 7
35. 0
37. \$27.09
39. −0.01

Section 24. Page 54

Problems:

1. -2
3. -59
5. 220
7. 51.95
9. -0.6294
11. $-\frac{1}{4}$
13. $-\frac{34}{35}$
15. $\frac{9}{11}$
17. $-17\frac{40}{69}$
19. $-96\frac{41}{70}$
21. $6{,}810.18$
23. $-2{,}222.22$
25. $305\frac{23}{48}$
27. $-61\frac{11}{24}$
29. 10
31. 4
33. $-10\frac{1}{3}$
35. 1.1

Section 25. Page 55

Problems:

1. $19x$
3. x
5. $-20w$
7. $13xz$
9. $-2cd$
11. $18M_x$
13. $-5.7x$
15. 0
17. $2x - 2y + 15$
19. $\frac{176}{5}y^3 + \frac{7}{2}x^2$
21. $4vd_x$
23. $6.9r - 11.4i$
25. $\frac{3}{11}P + \frac{3}{22}S$
27. $\frac{271}{8}P + \frac{1}{2}r + \frac{32}{5}$

Section 26. Page 59

Problems:

1. $6r - 4$
3. $20 + 2i$
5. $3P - 6r - 5$
7. $6P - 8$
9. $4 - 2a$
11. $10.2t - 2$
13. $2y - 6x - 9$
15. $y - x$
17. $4a^2 - 5ab - b^2$
19. $8p - 12m + n$
21. 15
23. $9r + 9n$
25. $4a + L$
27. $8x + 3$
29. $9 - 12a - 3b$
31. $-r - 3$
33. $2x^2 - (4x - 3y + 2)$
35. $K - (-P + Prt - D)$
37. $10 - (-1 - 1 - 1)$
39. $1 - (-3P + 4Q - R)$
41. $y^2 - y - (-6y^2 + 4x - 7)$
43. $(P + S) - (4r - 3n + 6q)$
45. $(2x + y) - (-5 + z - 3w)$
47. $-a - (b + c + d)$
49. $-(-7 + x - y)$

Section 27. Page 61

Problems:

1. x^8
3. $(1 + r)^2$
5. $6a^3b^5$
7. $(1.02)^6$
9. $a^5b^3c^4$

Page 62

1. x^2
3. $(1 + r)^2$
5. $a^2b^3c^2$
7. L^2R^5
9. $(8b^2)/9$

Page 62

1. x^4
3. x^6y^9
5. $(1.06)^9$
7. $16y^6$
9. a^3c^4
11. $(1.05)^{11}$
13. $343P^4$
15. $27r^3(1 + r)^5$

Page 63

1. 54
3. x^{15}
5. $54r^3i^3$
7. $8x^6$
9. x^3
11. $27yz$
13. $(1 + r)^5$
15. P^2r^3t
17. 2×3^2
19. 2^{17}
21. $3 \times 2^7 \times u^7v$
23. $ae(a + e)^6$
25. $S^3d^4t^2$
27. 2^{17}
29. ab^2c
31. w^{10}
33. $2^3p^3q^4$
35. $(a + e)^2$
37. $\dfrac{x^3y^3}{x^4 + y^4}$
39. $i^3 + j^3$
41. $X - 1$

Section 28. Page 65

Problems:

1. 144
3. -64
5. $-\frac{51}{32}$
7. -33
9. $-2a^2$
11. $14c^2$
13. $36z^4x$
15. PR^2t
17. $-10a^4y$
19. $81y$
21. $2x + 2y$
23. $2a^2 - 2ab$
25. $-2x - 2y + 6$
27. $ax + cx - 4x$
29. $dP - d^2P$
31. $A - Adt$
33. $x^2 - x^5$
35. $2x^2y + 2xy^2$
37. $\dfrac{p^4}{q^4} - p + 1$

Section 29. Page 66

Problems:

1. $a(1 + b)$
3. $3(a - 4b + 3c)$
5. $4ax(1 - 2x)$
7. $xy^2(3xyz^5 - 4)$
9. $2MN(1 - 4M + 10N^2)$
11. $4x^5y^2(x^2y^3 - 3)$
13. $0.04r^2t^5(t - r)$
15. $x(a + b)$
17. $P(1 - rt)$

Section 30. Page 67

Problems:

1. -4
3. $-\frac{4}{3}r$
5. $-\frac{3}{2}v^2$
7. $6bc^2$
9. $-(\frac{31}{9})m^3n^2$
11. $(1.08)^4$
13. $-5b$
15. -6.138
17. -1

Section 31. Page 71

Problems:

1. (*a*) $\frac{1}{2}$ (*c*) -3 (*e*) $\frac{5}{3}$ (*g*) $\frac{1}{2}$ (*i*) 100
2. (*a*) $4\frac{1}{4}$ (*c*) 0.854 (*e*) 156.06 (*g*) $\frac{3}{25}$ (*i*) 212.18 (*k*) 45 (*m*) 22 (*o*) $\frac{4}{7}$

3. $x = \frac{1}{2}$; $a = 1$
5. $a = \dfrac{S(1 - r)}{1 - r^n}$
7. $h = \dfrac{2A}{b}$; $b = \dfrac{2A}{h}$
9. 108.16
11. 0.045
13. \$7.50
15. A: \$875; B: \$125
17. 135 by 405 rods

Section 32. Page 74

Problems:

1. 2, $\frac{15}{7}$, $\frac{38}{5}$, $\frac{2}{3}$, 4, $\frac{11}{3}$, $\frac{26}{21}$, 9, 12
3. $\frac{4}{5}$
5. *A*, \$10; *B*, \$15
7. $\frac{1600}{147}$
9. 0.2, 0.24, 0.5, 0.06
11. 3.125

Section 33. Page 76

Problems:

1. (*a*) 14 (*c*) 17
3. $12\frac{1}{2}$ days
5. \$164.33
7. \$95.83
9. \$176.70
11. \$80, \$120
13. \$5,000, \$5,000, \$10,000
15. \$4

Chapter II. Page 77

Exercises:

1. (*a*) $10a$ (*c*) $5a + 2b$ (*e*) $2a^2 + 2ab - 2b^2$
2. (*a*) $7a$ (*c*) $8ay$ (*e*) $4p - 3q + r$
3. (*a*) $4a + 3b - 4c - 1$ (*c*) $7R - 6S - 1$ (*e*) $2x + y + 9$
4. (*a*) $15a^2b$ (*c*) $2PKrt$ (*e*) $8CD_x^2$
5. (*a*) $2b$ (*c*) $3z/a$ (*e*) $5.1N$
6. (*a*) 4 (*c*) -2 (*e*) -2
7. $P = I/(ni)$, $n = I/(Pi)$, $i = I/(Pn)$; 140

9. $P = S/(1+i)^n$; 1,000
11. $a = (2S - nL)/n$, $n = (2S)/(a+L)$, $L = (2S - na)/n$
13. (*a*) $x - 5$ (*c*) $(x + N)/2$ (*e*) $x - 20$
15. 200 lb
17. (*a*) $\frac{1}{6}$ days (*b*) $\frac{1}{13}$ days (*c*) $\frac{2}{5}$ days (*d*) $1/K$ days (*e*) $1/(m+n)$ days

19. 11 ft 8 in.
21. $4.46
23. 75

Chapter II. Page 79

Self-Test:

1. $(qH)/(30t)$
2. $x - 7$
3. $C = S - P - E$
4. $-r/15$
5. $(150rt)/(150 + r)$
6. $x^3/8$
7. $0.012z^2w^3 - 0.0048zw^3$
8. 4
9. $(mp + bq + cr)/(m + b + c)$
10. $K = (Pj - Cg)/(j - g)$
11. $D_x = \dfrac{D_{x+1} + D_{x+2} + D_{x+3}}{a_x - 1}$
12. $1.27
13. 2.21
14. 511 lb
15. 40

16. (*a*) -3 (*b*) 72 (*c*) 55,296 (*d*) $\frac{29}{24}$

17. *A*, $35; *B*, $58
18. $2\frac{1}{4}$ sq in.
19. $667.12
20. $2,400, $6,000, $9,600

Section 38. Page 95

Problems:

1. 14.45, 14.6
3. 7.42, 5.25
7. 68.20
9. 5.5

5. 79.9¢, $1.116, 78.4¢, 35.5¢

Section 39. Page 100

Problems:

1. (*a*) 7.7 (*b*) 7.625
3. 177.2 (for intervals 72.5–92.5, etc.; exact mean 174.9
5. $151.28

Section 40. Page 101

Problems:

1. 15
3. 82
5. 81.5
7. $5,400

Section 41. Page 103

Problems:

1. 5.75
3. 55.8
5. 6.5
7. $1.67
9. 123.6

Section 42. Page 107

Problems:

1. 20–25, 49.5–59.5
3. 58.6, 60, 60

Section 44. Page 110

Problems:

1. 0.094
3. Marksman B
5. 1.375
7. 2.65
9. 5.4
11. (*a*) 97.4; (*b*) 31.0

Section 45. Page 115

Problems:

1. 4.58
3. 17.5
5. 5.52
7. 5.5
9. 49
11. Mean, 2.5; S.D., 0.55

Section 46. Page 118

Problems:

1. $1.58, $1.74, $0.08
3. 2.2, 4.8, 6.2, 8.5
5. 37.3, 43.7, 65.2

Section 47. Page 120

Problems:

1. 79, 84, 81, 80, 91, 99, 100, 108, 118, 124
3. 73, 75, 74, 79, 86, 95, 100, 105, 112, 121

Chapter III. Page 121

Exercises:

1. Mean, $1.028; median, $1.021; modal group, $1–$1.03; Q_1, $0.962; Q_3, $1.09

3. 8.4, 8
5. 85.9
9. 2.9, 29

Chapter III. Page 124

Self-Test:

1. 80; 8.6
2. 18
3. 137.5
4. \$0.582; \$0.141
5. 227.625
6. 1,675
7. 145
8. 10
9. Mode
10. $Md = B + (n/C)L$

Section 48(1). Page 128

Problems:

1. (*a*) 6% (*c*) $12\frac{1}{2}$% (*e*) $16\frac{2}{3}$% (*g*) 4.5% (*i*) 125% (*k*) 550% (*m*) $11\frac{1}{9}$% (*o*) $66\frac{2}{3}$% (*q*) 100% (*s*) 1,000% (*u*) 7.638% (*w*) 92.357%

2. (*a*) 0.25, 25% (*c*) 0.3333, $33\frac{1}{3}$% (*e*) 0.2, 20% (*g*) 0.1667, $16\frac{2}{3}$% (*i*) 0.1111, $11\frac{1}{9}$% (*k*) 0.0909, 9.09% (*m*) 0.625, $62\frac{1}{2}$% (*o*) 0.875, 87.5% (*q*) 0.36, 36% (*s*) 1.9553, 195.53% (*u*) 2.625, 262.5% (*w*) 7.75, 775%

Section 48(2). Page 129

Problems:

1. (*a*) 0.03, $\frac{3}{100}$ (*c*) 0.0833, $\frac{1}{12}$ (*e*) 0.625, $\frac{5}{8}$ (*g*) 0.95, $\frac{19}{20}$ (*i*) 2.25, $\frac{9}{4}$ (*k*) 0.005, $\frac{1}{200}$ (*m*) 4.35, $\frac{87}{20}$ (*o*) 10, 10 (*q*) 0.005, $\frac{1}{200}$ (*s*) 0.004, $\frac{1}{2500}$

2. (*a*) 0.75, 75% (*c*) $\frac{1}{40}$, 2.5% (*e*) 0.7143, 71.43% (*g*) 4.875, 487.5% (*i*) 2.6, $\frac{13}{5}$ (*k*) $\frac{199}{2}$, 9,950%

Section 49(1). Page 130

Problems:

3. (*a*) \$18 (*c*) \$600 (*e*) 17.50 (*g*) 3.4 days (*i*) \$21.60 (*k*) \$1,662.50 (*m*) \$818.86

5. \$3,520

Section 49(2). Page 132

Problems:

1. (*a*) 50% (*c*) 20% (*e*) 6% (*g*) 3.5% (*i*) 12.5% (*k*) 200% (*m*) 33.86% (*o*) 2.9%

2. (*a*) 10% (*c*) 18.75% (*e*) 40% (*g*) 2% (*i*) 4.5% (*k*) 21.43% (*m*) 8.33% (*o*) 133.33%

3. $5\frac{1}{2}$%
5. 16.75%
7. 12%
9. 11.54%
11. $4\frac{1}{2}$%

Section 49(3). Page 133

Problems:

1. 100
3. 40
5. 35
7. \$65
9. \$500
11. \$80,000
13. \$203,000,000, \$227,000,000
15. \$4,050

Section 50. Page 135

Problems:

1. \$23.25, \$441.75
3. 2.78%
5. \$36,240
7. \$2,743.10; \$1,987.75; \$5,554.00
9. \$550, $56\frac{1}{4}\%$

Section 51. Page 138

Problems:

1. (*a*) \$100.80 (*c*) \$463.23 (*e*) \$2,625 (*g*) \$353.46

3. \$44.10; 20%

5. (*c*)

7. \$653.56, \$660.23
11. (*a*) 35.2% (*c*) 56.8% (*e*) $r[6 - (11r)/100 + (6r^2)/10{,}000]\%$

13. Loss, \$33.46

15. \$29.18

Section 52. Page 141

Problems:

1. \$31.60
3. (*a*) \$35 (*c*) \$122.50
4. (*a*) $12\frac{1}{2}$ (*c*) 4
5. (*a*) \$7,500 (*c*) \$22,857.14

7. \$763.34
8. (*a*) 3¢ (*c*) 5¢ (*e*) none (*g*) 2¢ (*i*) \$66 (*k*) 25¢ (*m*) 40¢ (*o*) none
9. (*a*) \$4 (*c*) \$0.80 (*e*) \$2.95 (*g*) \$1.96 (*i*) \$0.43 (*k*) \$1.42 (*m*) \$392.99 (*o*) \$30.69

Section 53. Page 144

Problems:

1. (*a*) \$6.12 (*c*) \$62.94 (*e*) \$7.93

3. \$39.22
5. \$7.40; \$11; \$36
7. \$11; \$19; \$80
9. \$0.74; \$2.94; \$2.94
11. \$3; \$10; \$13.33
13. 42.22%
15. 60%
17. \$69.23; \$24.23
19. 23.79%
21. (*a*) 1.54 (*c*) 1.18

Chapter IV. Page 146

Exercises:

1. (*a*) 0.0125 (*c*) $13\frac{1}{3}\%$ (*e*) $r/100$
3. $31\frac{1}{4}\%$
5. Second, \$95
6. (*a*) 20% (*c*) $33\frac{1}{3}\%$ (*e*) $33\frac{1}{3}\%$
7. (*a*) \$324.90 (*c*) \$1,411.34
9. (*a*) 8.74% (*c*) 14.29%
11. 80.32%, 12.05%, 6.02%, 1.61%
13. \$7,189.00
15. \$2,740
17. \$1,172
19. \$88
21. \$4,772.73
23. \$143,286.27
25. \$1,200
27. \$0.36, \$0.77, \$2.94
29. \$1.94
31. \$64.29
33. \$239.40, \$1.81, 47.01%
35. 293

Chapter IV. Page 150

Self-Test:

1. 920
2. 39.1%
3. 62.5%
4. 20%
5. \$350
6. 37.5%
7. $\left(\frac{100D - C}{D}\right)\%$
8. \$47.62
9. $133\frac{1}{3}\%$
10. \$7
11. \$145.80
12. 14.5%
13. 90
14. 0.0235
15. $\frac{9}{50}$
16. 15%
17. 200%
18. $K/100$
19. 36,500%
20. 27.1%

Section 54. Page 154

Problems:

1. (*a*) \$105.30, \$885.30 (*c*) 0.04, \$2,040 (*e*) \$648, 9 yr
3. 3 yr ago
5. 24%
7. \$1,500
9. 100%
11. \$0.29, \$100.29
13. 5.28%

Section 56. Page 158

Problems:

1. (*a*) 194, 190 (*c*) 200, 196 (*e*) 397, 390 (*g*) 367, 361
2. (*a*) \$1.45, \$1.43 (*c*) \$0.83, \$0.82 (*e*) \$7.38, \$7.28 (*g*) \$57.99, \$57.19

3. \$90.42
5. \$162.50
7. \$5.08, \$5

9. Exact interest for approximate time, \$1

11. \$91.33

13. \$150

Section 57. Page 160

Problems:

1. (*a*) \$320 (*c*) \$82.13 (*e*) \$124.83 (*g*) \$223.30 (*i*) \$900

3. \$738.92
5. Second; \$0.50
7. \$186.73
9. \$48,828.12

Section 58. Page 162

Problems:

1. (*a*) \$28.30 (*c*) \$43.81 (*e*) \$31.92 (*g*) \$4 (*i*) \$1.52

2. (*a*) \$1.68 (*c*) \$5.44 (*e*) \$2.02
3. \$7.46

Section 60. Page 165

Problems:

1. \$1.01, \$301.24
3. \$7.22, \$955.44
5. \$1.01, \$150.30
7. \$314.58, \$50,018.75
9. \$37.78, \$4,999.72
11. \$29.18, \$2,596.82
13. \$6.56, \$679.67
15. \$0.75, \$150
17. \$5.02, \$999.98
19. 2.98%
21. 120 days
23. \$3.38

Page 167

1. \$1.75, \$198.25
3. \$7.91, \$442.09
5. \$0.41

Section 61. Page 170

Problems:

1. 4.98%
3. (*a*) 2.99% (*b*) 2.97%
5. (*a*) \$17.00 (*b*) \$16.67
7. Bank discount; \$4.69
9. \$4,050.63

Section 62. Page 174

Problems:

1. 21.2%
3. 43.9%
5. 10.6%
7. 18.9%
9. 1.03%; 1.01%
11. 21.5%

Section 63. Page 176

Problems:

1. \$594.03, \$593.04
3. \$26.68, \$26.67
5. $[P(1 + rt) - P_1][r(T - t) + 1]$
7. $Pr^2t(T - t),\ t = 0,\ t = T$

Chapter V. Page 177

Exercises:

1. \$168
3. 5.45 yr
5. \$83.57
9. \$396.50, \$405.17
11. 16%
13. \$348.98
15. \$1,408.33, \$1,308.33, \$1,283.33
17. \$128.74
19. \$404.04
21. Dec. 21, 1962
23. \$1,480; 4.05%
25. \$30,132.81
27. \$10,050.25
28. (*a*) \$1.32 (*c*) \$0.26 (*e*) \$2.50
29. \$305.88
31. \$11.52
33. \$307.50
35. 19.2%

Chapter V. Page 180

Self-Test:

1. \$11.84
2. 1.02%
3. $2\frac{1}{2}$ yr
4. 217
5. 104
6. 44
7, 8, 9. 215, 102, 44
10. \$7.92
11. \$485.44
12. 23.6%
13, 14. \$508, \$508.03
15, 16. \$10; \$990
17, 18. \$1,616, \$4.04
19, 20. \$999.90, \$997.40

Section 64. Page 182

Problems:

1. $9x + 3y$
3. $8x^2y - 7y$
5. $6x^2 - 6y^2$
7. $15 - 7x^2y$
9. $6y^3 - 10y^2 + y + 114$
11. $3a$
13. $-10i$
15. $2a + b$
17. $2x - 13y + 5w$
19. $7y^2 - x^3 - 7$
21. $19 - 6t + 4r - 9rt^2$
23. $\frac{9}{8}R - \frac{7}{6}P - \frac{1}{10}Q$
25. $-3a - 4b - 7c$
27. $10x + xy - 3 + 2y$

Section 65. Page 183

Problems:

1. $a^2 - b^2$
3. $x^2 + 3xy + 2y^2$
5. $4r^2 + 4rd + d^2$
7. $F^2 + F - 12$
9. $C^2 + C - 6$
11. $a^2 + 2aL + L^2$
13. $18c^2 + cd - 4d^2$
15. $20a^4 + 3a^2b - 2b^2$
17. $i^4 - 2i^3r - 8ir^2 + 16r^3$
19. $x^3 - y^3$
21. $c^3 + 3c^2d - cd - 3d^2$
23. $a^3 + 3a^2b + 3ab^2 + b^3$

25. $ax^3 - 3x^4 + 2yx^3 + ay^3 - 3xy^3 + 2y^4$

27. $m^5 - n^5$
29. $a^2 + 2ab + b^2 - a^2b^2$
31. $6a^4 - 7a^3 + a^2 + 5a - 2$
33. $1 + 2x + x^2 - 2y - 2xy - 3y^2$
35. $P^3 + 3P^2S + 3PS^2 + S^3$

37. $0.245p^4 + 0.35p^3 - 0.463p^2 + 0.588p - 0.588$

Section 66. Page 185

Problems:

1. $3ab(c - 9)$
3. $(x + y)(x - y)$
5. $(x + 3)^2$
7. $(ab + 5mn)(ab - 5mn)$
9. $(2a + 1)(a + 2)$
11. $(8z - 3w)^2$
13. $2(5F + 6N)(5F - 6N)$
15. $3c(a^2 + b^2)(a + b)(a - b)$
17. $(15a - 14p)(2a + 3p)$
19. $(p + q + 1)(p + q - 1)$
21. $cd(c - d)$
23. $2y(z^2 - 7zy + 4)$
25. $(2.03 + r)(r - 0.03)$
27. $t^{n-2}(Adt^2 + 1)$
29. $9p(4q^2 + 1)^2$

Section 67. Page 188

Problems:

1. $\dfrac{12 + 24r}{(1 + r)^2}$
3. $\dfrac{2aby - 3ax}{x^2y^2}$
5. $-\frac{89}{28}W$
7. $\dfrac{14x^3 + 105x + 27}{63x^2}$
9. $\dfrac{0.96P - 0.04}{P + 1}$
11. $\dfrac{4P - 2SRP + 5S^2R}{6R^2S}$
13. $\dfrac{37pq - 2p^2 - 3q^2}{6pq}$
15. $\dfrac{16y^3 - 8xy^2 + x^2y - 4y^2 - 6xy}{(x + 3y)(x - 4y)^2}$
17. $\dfrac{5k + 2i}{(2i)(k - 2)}$
19. $\dfrac{d(t - 1)}{d^2t^2 - 1}$

Section 68. Page 189

Problems:

1. $\dfrac{a^5bc^3}{x^2y^5}$
3. $\dfrac{5mrs}{2n}$
5. $\dfrac{10zda^5}{cw}$
7. $\dfrac{x}{y}$
9. $\dfrac{a^2}{3b}$
11. $(x + y)(2x - 3)$
13. $\dfrac{1}{x^2}$
15. $\dfrac{3x^2 - 11x + 6}{x - 2}$
17. $\dfrac{i + r}{i^2r^3}$
19. $p(q + p - 2)$
21. $\dfrac{A}{(1 + r)^2}$
23. $200(C + P)$
25. $-4d^2i^2$
27. $\dfrac{-k}{3m(k + m)}$

Section 69. Page 191

Problems:

1. (a) $\sqrt{50}$ (c) $\sqrt{16x}$ (e) $\sqrt[3]{xy}$ (g) $\sqrt[4]{1+i}$ (i) $\sqrt[5]{(5r)^3}$ (k) $\sqrt[3]{(6x^2y^3)^2}$ (m) $14\sqrt[4]{x^3y^3}$ (o) $\sqrt[8]{(18p^3q^2)^7}\sqrt[5]{(1.04)^3}$

2. (a) $3ab^{\frac{1}{3}}$ (c) $-9m^{\frac{9}{2}}$ (e) $(1+r)^{\frac{1}{3}}$ (g) $(7^3a)^{\frac{1}{5}}$ (i) $(RS)^{\frac{1}{3}}T^{\frac{3}{4}}$ (k) $(a^3-x^3)^{\frac{1}{3}}$ (m) $(S^{\frac{1}{3}})^{\frac{1}{5}}$ (o) $(d^{12}q^{\frac{8}{3}})^{\frac{1}{4}}$

3. (a) $a^{\frac{3}{2}}$ (c) $3s$ (e) 2 (g) $x^{\frac{1}{4}}y^{\frac{11}{4}}z^3$ (i) $r^{\frac{13}{30}}S^{\frac{7}{60}}$ (k) $2a(1-a)^{\frac{1}{3}}$ (m) $(1+r)^2$ (o) $(1.04)^2$ (q) $2\frac{26}{27}$ (s) $-1{,}562.4$

Section 70. Page 193

Problems:

1. 68
3. $(1+i)^6$
5. 2
7. 1
9. $(1.08)^6$
11. 1

Section 71. Page 194

Problems:

1. 9, 64/27, 1/729, 1/1.44

2. (a) $\left(\dfrac{1}{1.06}\right)^5$ (c) $\dfrac{1}{(1+r)^{15}}$ (e) $(a+b)^3$ (g) $\dfrac{P}{(1+R)^{11}}$ (i) $\dfrac{y^4}{x^7z}$ (k) $\dfrac{q^2(1+r)^2}{p^2}$ (m) $5^{\frac{1}{2}}$ (o) $\dfrac{100}{(1+r)^4}$

3. $(x/y)^{\frac{1}{3}}$

5. $\dfrac{a^7}{c^2b^5}$

7. $\dfrac{1}{3^{\frac{1}{2}}RS^{\frac{1}{3}}}$

9. $\dfrac{1}{x^4y^{\frac{3}{2}}}$

11. $(j/i)^{\frac{1}{2}}k^{\frac{n}{2}-1}$

13. $\dfrac{r-S}{r^2S^2}$

15. $\dfrac{(u+r)(u+2r)}{(ru)^2}$

17. $R-S$

19. 1.06

21. $-\frac{3}{2}$

23. -400

25. $-0.05/(1.05)^2$

27. -1

Section 72. Page 196

Problems:

1. 361
3. 20,736
5. 804,357
7. 50.653
9. 3.125×10^{-7}
11. 14,400
13. 23
15. 0.71
17. 113,969.25
19. 0.77
21. 12.12
23. 20.12
25. $2\sqrt{2}$
27. $2\sqrt[3]{3}$
29. $7ir^2\sqrt{2i}$
31. $4S^2\sqrt[3]{2St^2}$
33. $0.5p^3s^5\sqrt{3pq}$
35. $\sqrt{6}$
37. $\sqrt{10xy}$
39. $z^3\sqrt{3y}$
41. $10T\sqrt{ST}$
43. $q^2\sqrt{0.2q}$
45. $(1.05)^2p\sqrt[6]{p}$

Page 198

1. (*a*) 27 (*c*) $\frac{1}{8}$ (*e*) 9 (*g*) 0.0609 (*i*) 7
2. (*a*) 583.2 (*c*) 406 (*e*) 189 (*g*) 0 (*i*) 1.03 (*k*) 0.31

Section 73. Page 201

Problems:

1. 18.854
3. 19.939
5. 1.040
7. 0.254
9. 0.060
11. 30.6
13. 1.4
15. 2

Chapter VI. Page 205

Exercises:

1. (*a*) $5L_x - 4L_{x+1} + 6L_{x+2}$ (*c*) $4an - nL + 15$ (*e*) $4xy - 2x^2y + 3xy^2$
2. (*a*) $3x - y$ (*c*) $\frac{2}{3}x_1y_1 + \frac{2}{3}x_2y_2 + \frac{2}{3}x_3y_3$ (*e*) 1

3. $9x^2 - 4$
5. $R^3 - S^3$
7. $2P^3 - 38P^2 - 44P + 80$
9. $4(7a + 5)(a - 1)$
11. $a^2b^2(c + d)(c - d)$
13. $x - 7$

15. (*a*) $12x^2 - 3x$ (*c*) $r^2 - r - 20$ (*e*) $27s^3 - 12s$
16. (*a*) $P(1 + ni)$ (*c*) $(d - 3)(d + 2)$ (*e*) $3m(m - 4)(m + 2)$ (*g*) $C(1 - r)^2$
17. (*a*) $\dfrac{2x + 3y}{-2}$ (*c*) $\dfrac{P + C}{2 + 4P}$ (*e*) $a^2 - b^2$

18. (*a*) $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ (*c*) $1 + 3R + 3R^2 + R^3$
(*e*) $16x^4 - 32x^3 + 24x^2 - 8x + 1$

19. (*a*) $\frac{2x - 5}{x}$ (*c*) $\frac{1 - r + r^2}{r^4}$ (*e*) $\frac{x}{(x^2 - 1)}$

29. (*a*) x^4y^6 (*c*) 2 (*e*) $-24x^3y^5$ (*g*) 6^{2n} (*i*) $\left(\frac{cd}{2}\right)^{\frac{1}{3}}$ (*k*) $\frac{2}{a^5b^3}$

30. (*a*) 70,000 (*c*) $\frac{12}{11}\sqrt{11}$ (*e*) $14\sqrt{6}$

31. (*a*) 8.71 (*c*) 32.04 (*e*) 0.38 (*g*) 14.21 (*i*) 3.00

Chapter VI. Page 208

Self-Test:

1. $5m - 7n$
2. $N(L_x + L_{x+1})(D + 2)$ dollars
3. $1 - b - c + bc - a + ab + ac - abc$
4. $n(a + L)$
5. (*a*) $-6x$ (*b*) $2a^{\frac{1}{3}}b^{\frac{1}{3}}$ (*c*) $\frac{b}{a^4}$ (*d*) $\frac{2}{a^3b^2}$
6. (*a*) $-\sqrt{3}$ (*b*) 2 (*c*) $3xy^2$
7. $1 + \frac{2j}{m} + \frac{j^2}{m^2}$
8. $\frac{2}{3}R$
9. $\frac{100rt}{100 + k}$
10. $\frac{P + S - R}{R^2}$
11. $\frac{9 + 3x + x^2}{3x}$
12. $m(m^2 - n^2)$
13. $2x + 5$
14. $H^3 + 3H^2T + 3HT^2 + T^3$
15. $x^2 + y^2 + r^2 + 2xy + 2xr + 2yr$
16. 7, 23
17. Between $t = 2$ and $t = 4$
18. $-\frac{7}{2}$
19. Increase
20. 48.58

Section 75. Page 213

Problems:

1. (*a*) $\log 10000 = 4$ (*c*) $\log 87.55 = 1.94226$ (*e*) $\log 35 = 1.54407$
2. (*a*) $5 = 10^{0.69897}$ (*c*) $290 = 10^{2.46240}$ (*e*) $5.25 = 10^{0.72016}$
3. (*a*) 5 (*c*) 3 (*e*) 3 (*g*) 10 (*i*) $\frac{1}{16}$

Section 76. Page 215

Problems:

1. 0 **3.** 1 **5.** 0 **7.** 2 **9.** 5 **11.** 0 **13.** 0 **15.** 0

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1. $9 - 10; -1$
3. $9 - 10; -1$
5. $8 - 10; -2$
7. $8 - 10; -2$
9. $7 - 10; -3$
11. $7 - 10; -3$
13. $6 - 10; -4$
15. $6 - 10; -4$
17. $5 - 10; -5$
19. $5 - 10; -5$

Section 77. Page 218

Problems:

2. (*a*) 2.77866 (*c*) 3.77815 (*e*) 3.78097 (*g*) 7.77887 − 10 (*i*) 5.78089 − 10

3. (*a*) 2.95036 (*c*) 1.57634 (*e*) 8.74966 − 10 (*g*) 3.53782 (*i*) 0.99782 (*k*) 2.01368 (*m*) 1.11628 (*o*) 9.86320 − 10

Section 78. Page 219

Problems:

1. 60.07 **3.** 6011 **5.** 0.06015 **7.** 36 **9.** 2195 **11.** 0.005429
13. 467,800 **15.** 1.448

Section 79. Page 220

Problems:

1. (*a*) 4.77853 (*c*) 4.77857 (*e*) 1.77857
2. (*a*) 9.67728 − 10 (*c*) 7.67731 − 10 (*e*) 4.67724

Page 222

1. 0.99185, 0.99185, 0.99185 **4.** (*a*) 24.031 (*c*) 0.57565
3. (*a*) 2,024.5 (*c*) 0.20247

Section 80. Page 224

Problems:

1. (*a*) 5.62881 (*c*) 7.61470 − 10 (*e*) 5.41019
2. (*a*) −8,096 (*c*) −7.1682
3. (*a*) 1,728, 0 (*c*) 36,068, 0.32 (*e*) 1,499.2, 0.03409 (*g*) 1,695,100, 35.60
4. (*a*) 135.16 (*c*) 0.030011 (*e*) 1.2679×10^{9} (*g*) 123,590 (*i*) 7.6910×10^{8}

Page 227

1. (*a*) 2.12413 (*c*) 8.50982 − 10 (*e*) 7.52159 − 10
2. (*a*) 15.004 (*c*) 10.393
3. (*a*) 8.6143 (*c*) 0.90548 (*e*) 3.1734×10^{-7} (*g*) 4.6284 (*i*) −0.059961
4. (*a*) −37.258 (*c*) 31.849 (*e*) 1,559,300 (*g*) 1.2346 (*i*) 5.0318×10^{-6}

Page 229

1. (*a*) 6.49436; 3,121,500 (*c*) 9.52870 − 10; 0.33783 (*e*) 6.19074 − 50; $1.56(10^{-44})$
3. (*a*) 1.3685 (*c*) 23,730 (*e*) 0.79035

4. (*a*) 170.30 (*c*) -9210.4 (*e*) 974.30 (*g*) 16,782 (*i*) 334.32 (*k*) 1.5259×10^{-5} (*m*) 118.51 (*o*) 451,690

Page 231

1. (*a*) 0.89107 (*c*) 0.06090 (*e*) $9.61032 - 10$ (*g*) $9.60025 - 10$ (*i*) $9.67210 - 10$

2. (*a*) 21.030 (*c*) 0.45258 (*e*) 10.150 (*g*) 3.1591 (*i*) 29,091 (*k*) 1.6164 (*m*) 0.98322 (*o*) 1.2347

4. (*a*) 70,482 (*c*) 4.0940

Chapter VII. Page 237

Exercises:

1. (*a*) 29,111 (*c*) 1,032.4 (*e*) 0.14319 (*g*) 597.41 (*i*) -3.0001 (*k*) 1.7705 (*m*) 1.7768 (*o*) 0.57687

5. 343,450,000
7. 3
9. 28
11. 2.38
13. 0.0137
15. -0.71903
17. 2.9439
19. -7.9433

Chapter VII. Page 238

Self-Test:

1. 2.74974
2. $8.74896 - 10$
3. $8.75097 - 10$
4. 0.87533
5. 1.75032
6. $6.74860 - 10$
7. 5.49947
8. 3.00178
9. 3.18607
10. 2.14
11. 561.15
12. 0.056264
13. 0.056354
14. 56,020,000
15. 5.612
16. 1.0083
17. 0.13045
18. 134.98
19. 9,959.0
20. 0.0306

Section 82. Page 245

Exercises:

1. 79,486
3. -33, -60
5. -119, -605
7. 19.9, 505
9. $2\frac{2}{3}$, $22\frac{1}{2}$
11. 10, $25\frac{5}{6}$
13. -2, 17
15. 5, $\frac{9}{2}$
17. $\frac{1}{8}$
19. 26, 3
21. (*a*) 13, 20, 27, . . .
(*c*) -8, $-3\frac{3}{7}$, $1\frac{1}{7}$, . . .
23. \$23,625
25. $190\frac{1}{2}$ in.
27. 5%
29. \$250
31. \$201
33. 24 ft

Section 83. Page 255

Exercises:

1. 81,121
3. 640, 1,275
5. 405, 605
7. 12,288, 9,831
9. 54/2,401, 8,282/7,203
11. 88,064, 175,956
13. 1/128, −85/128
15. $1{,}215\sqrt{3}$, $\dfrac{3{,}640}{\sqrt{3} - 1}$
17. 3, 605
19. 4, 0.042
21. $3\frac{1}{4}$
23. 1.9549, 1,307.1
25. 32.000, 3/22,768
27. (*a*) $a = \dfrac{(r - 1)S}{r^n - 1}$
29. $\frac{3}{11}$
31. $\frac{1}{3}$
33. 256
35. 120%, 3,300,000
37. 3.1%, 1.9%
39. 11.99 ft

Section 86. Page 264

Exercises:

1. (*a*) $109.34 (*c*) $249.99
3. $1,118.29
5. $5,361.60
7. 4 yr
9. $2\frac{1}{4}$%; 2.243%
11. $345.03
12. (*a*) $1,302.72 (*c*) $348.16
13. $1,100 due in 5 yr
14. (*a*) 0.85% (*c*) 17.46 yr

Section 87. Page 266

Problems:

1. (*a*) $366.06 (*b*) $565.56 (*c*) $776.70
2. (*a*) $889 (*b*) $887.97 (*c*) $887.45
3. $861.67; $742.47
5. $8,203.48; $7,440.94; $6,755.64
7. 0.861667; 0.622750; 0.975371

Section 88. Page 268

Problems:

1. (*a*) 2.01% (*c*) 4.03%
2. (*a*) 2.0% (*c*) 4.0%
3. $12[\sqrt[12]{(100 + K)/100} - 1]\%$
5. 2.51%

Section 89. Page 272

Problems:

1. $25,109.31
3. $R = \dfrac{rS}{(1 + r)^n - 1}$
5. $9,419.24
7. $8,831.77
9. $23,697.51
11. $4,115.67
13. Approximately 84

Section 90. Page 275

Problems:

1. \$190.39
3. Cash, \$236.24
5. \$3,444.69
7. Approximately \$15,000
9. 36.5%

Section 91. Page 279

Problems:

1. \$461.95
3. \$870.17
5. \$510.19
7. \$847.35
9. \$3,421.30

Section 92. Page 283

Problems:

1. 5.013298
2. (*a*) 9.949875 (*c*) 1.04875 (*e*) 3.87695
3. (*a*) 1.001992 (*b*) 1.0192
4. (*a*) $1 - 2x + 3x^2 - 4x^3$ (*c*) $1 - \frac{r}{2} + \frac{3r^2}{8} - \frac{5r^3}{16}$
(*e*) $1 + 2R + 3R^2 + 4R^3$
5. 0.44444, 0.25
7. 9.2096, 9.214226
8. (*a*) $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab + b^7$
(*c*) $1 + 5(0.02) + 10(0.02)^2 + 10(0.02)^3 + 5(0.02)^4 + (0.02)^5$
(*e*) $x^7 - 14x^6y + 84x^5y^2 - 280x^4y^3 + 560x^3y^4 - 672x^2y^5 + 448xy^6 - 128y^7$
(*g*) $a^{16} - 8a^{14}b^2 + 28a^{12}b^4 - 56a^{10}b^6 + 70a^8b^8 - 56a^6b^{10} + 28a^4b^{12} - 8a^2b^{14} + b^{16}$
9. \$444.07
10. (*a*) 10.049875 (*c*) 5.0661 (*e*) 3.018347 (*g*) 2.080102
11. $i = (1 + j/m)^m - 1$
12. (*a*) 1,030,301 (*c*) 941,192 (*e*) 132,651 (*g*) 9,409
13. (*a*) $11ab^{10}$ (*c*) $34r^{16}$ (*e*) $(-30x)(y/2)^9$

Chapter VIII. Page 286

Exercises:

1. (*a*) 75, 390 (*c*) −43, −329
3. \$405
5. \$1,220
6. (*a*) 2 (*c*) $-\frac{1}{4}$ (*e*) r^2 (*g*) $(1.04)^2$ (*i*) $(1 + i)^2$
7. (*a*) 10.5828 (*c*) 17.25786
8. (*a*) 1.0609 (*c*) $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
(*e*) 1.265319 (*g*) $1 - 10z + 45z^2 - 120z^3 + \cdots + z^{10}$
(*i*) $0.0001p^2 - 0.008p^{\frac{3}{2}} + 0.24p - 3.2p^{\frac{1}{2}} + 16$
9. (*a*) 7.141429 (*c*) 12.041595

10. (*a*) \$1,428.25; \$428.25 (*c*) \$3,811.47; \$811.47 (*e*) \$5,649.50; \$2,149.50 (*g*) \$2,980.70; \$380.70

11. (*a*) \$819.54 (*c*) \$441.79
13. 5.9%
15. \$3; \$3.01
17. 0.058
19. \$476.89
21. \$84,000
23. \$3,400, \$31,900
25. $2\frac{5}{44}$ miles
27. 49,500 yd
29. 24.64%; 9,348,000
31. 7.45 yr
32. (*a*) \$6,706.04 (*c*) 8.58 yr

Chapter VIII. Page 289

Self-Test:

1. 19
2. 69
3. 16
4. 32/125
5. 2,062/125
6. 255/128
7. $1/2^{n-1}$
8. 2
9. 5.98
10. $126p^5q^4$
11, 12. 5.013
13, 14. \$300.25
15, 16. 25 yr
17, 18. 1.0025%
19, 20. \$672.16

Section 94. Page 291

Problems:

1. $S = \dfrac{R(1+i)^n - 1}{i}$

3. $S = C - nD$

5. 3

7. $P_x = {}_nE_x/v^n$

9. $M_x = -4.2$

11. $M_x = P_xN_x$

13. $i = \dfrac{1-(1.03)^{-n}}{a_{\overline{n}|i}}$

15. $x = \dfrac{M_1n_1 + M_2n_2 + \cdots + M_rn_r}{M_1 + M_2 + \cdots + M_r}$

17. $h = \dfrac{pf}{1-p}$

19. $n = \dfrac{\Sigma d_1d_2}{r\sigma_1\sigma_2}$

21. $x = \dfrac{6}{1+2t}, t = \dfrac{6-x}{2x}$

23. $b_1 = \dfrac{2A - b_2h}{h}$

25. $A = \dfrac{A_1}{v^2} + \dfrac{A_2}{v} + A_3$

27. $w = \dfrac{2u - 3v}{u - v + 3}$

29. $1\frac{2}{23}$

31. $\frac{5}{17}$

33. 7

35. \$2,000 @ 1%; \$3,000 @ $2\frac{1}{2}$%

37. $66\frac{2}{3}$ lb and $33\frac{1}{3}$ lb, respectively

39. $\frac{21}{29}$

41. $3\frac{3}{7}$ min

Section 95. Page 297

Problems:

1. $x = 2, y = 4$
3. $P = 300, r = -0.94$
5. $x = 1, y = 2, z = 3$
7. $x = \frac{18}{11}, y = \frac{31}{11}, z = \frac{83}{11}$
9. $x = 4, y = -1$
11. $x = 4, y = 4, z = 4$
13. $x = 4, y = -2, z = 1$
15. $x = -1, y = 1, z = 4$
17. $w = 2, z = -3$
19. $x = -2, y = -5$
21. $x = 1, y = 2$
23. $x = -2, y = 3$
25. 40, 60
27. $8,000 assets, $2,000 liabilities
29. $21,100 @ 3%, $8,900 @ $3\frac{1}{2}$%
31. $1,650.74 salary, $1,841.90 tax
33. $795.18 bonus, $530.12 tax
35. $100, 6 months, 9 months
37. Si, 5 sheep; Alf, 7 sheep
39. A, $1,000; B, $1,500, C, $900

Section 96. Page 301

Problems:

1. $x = -8, +1$
3. $x = (3 \pm \sqrt{5})/2$
5. $x = 5, -7$
7. 5 in. × 10 in.
9. 2 ft
11. 16.33%
13. 40 ft × 140 ft

Section 97. Page 304

Problem:

3. $4,493

Page 305

Exercises:

1. 8
3. 4.71
5. 2
7. $n = \dfrac{\log L - \log a}{\log r} + 1$
9. 4.68
11. 1960
13. $1,349.90

Chapter IX. Page 312

Exercises:

1. $1,088.20
3. $130 cash; $3.67
5. 0.45 yr
7. 90 days hence
9. 51.1%
11. 1.64%
13. $12,485
15. 31.7 months
17. 209.18
19. 104.4 days
21. 5.27 yr

23. (*a*) $477.68, $1,014 (*b*) 34 days hence
25. 6 qts

27. 20 sheep, 18 pigs, 6 cows
29. (*a*) $x = 15$, $y = 9$ (*c*) $x = 150$, $y = 5$ (*e*) $x = 16$, $y = 7$

31. 6.6 yr
33. \$3,654.43
35. \$13,590.33
37. \$586.15
39. 8.8 yr

Chapter IX. Page 315

Self-Test:

1. 12.2 yr
2. \$2,668.04
3. 2.44 yr hence
4. $15\frac{5}{8}$ lb @ \$1.80, $9\frac{3}{8}$ lb @ \$1
5. 2.02
6. $x = 3$, $y = 5$
7. $T = 2.5, 3$
8. \$1,530.34
9. 12
10. $R = \dfrac{Pr}{1 - (1 + r)^{-n}}$

Section 99. Page 320

Problems:

1. 6
3. 24
5. 7, 60, n
7. 120
9. 60
11. 1,260
13. 60
15. 2,450,448
17. 210
19. 22,308
21. 66
23. 81,796

Section 100. Page 323

Problems:

1. 5/8
3. 4/9
5. 1/3
7. 8/25
9. False
11. 1/2

Page 326

1. 0.99712
3. $\dfrac{l_{x+1}}{l_x}$
5. 0.99065
7. 0.40124
9. 0.029
11. 0.002

Section 101. Page 328

Problems:

1. \$2,916.67
3. \$1,402.16
5. \$0.25
7. \$426.72
9. \$990.39
11. \$0.04
13. \$3,750

Section 102. Page 330

Problems:

1. 1/8
3. 0.058
5. 1/64
7. $1,178.27

Page 332

1. 0.002
3. 3/10, 9/25
5. 0.0179

Page 333

1. 5/18
3. 11/20
5. 1/2
7. 0.907
9. 5/7

Section 104. Page 339

Problems:

1. *(a)* 0.3125 *(b)* 0.6875 *(c)* 0.875
3. *(a)* 1; 0.913 *(b)* 3.6; 0.6 *(c)* 3; 1.225 *(d)* $1\frac{3}{4}$; 1.146

5. 0.008
7. 0.081; 0.0005

Section 105. Page 346

Problems:

1. *(a)* 0.3413 *(c)* 0.4773 *(e)* 0.6826 *(g)* 0.1934 *(i)* 0.3669

2. *(a)* 0.207 *(c)* 0.666
3. 0.034
5. 27
7. 10.12

9. *(a)* About 151 *(b)* About 135 *(c)* About 69 *(d)* $0.80–$0.94

Section 106. Page 352

Problems:

1. 0.067; no
3. 0.002
5. Yes; $t = 2.33$

Chapter X. Page 353

Self-Test:

1. 120
2. 1,140
3. $1/(47)^3$
4. 0.933
5. 0.999
6. 1/18
7. 8/27
8. About 34
9. 96.1–143.9
10. Yes; $t = -5$

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